16.2) Line Integrals

Say we have a curve C of finite length in either the x,y plane or x,y,z space. We will focus on the former situation, for the sake of simplicity, but the concepts we are about to develop can be applied to the latter as well.

The curve C could be a line segment. (We consider a line to be a special case of a curve–i.e., it is a *straight* curve.)

Suppose C has been parameterized with the parametric equations $x = x(t)$, $y = y(t)$, where $t \in [a,b]$. We typically think of t as representing *time*. This parameterization dictates a *positive direction* on the curve, which is the direction one moves along the curve as t increases. The curve is thus referred to as an **oriented curve**. In contrast, a curve without a parameterization is a **non-oriented curve**.

Let P_a denote the point $(x(a),y(a))$, and let P_b denote the point $(x(b),y(b))$. We refer to the former as the curve's **initial point** and to the latter as the curve's **final point**.

C is said to be a **closed curve** if $P_a = P_b$.

For example, in the x,y plane, the unit circle centered at the origin is $x^2 + y^2 = 1$. Expressed in this form, it is a non-oriented curve. It may be parameterized as $x = \cos t$, $y = \sin t$, with $t \in [0,2\pi]$, in which case it is an oriented curve, with its positive direction being counter-clockwise. It is also a closed curve, with $P_0 = P_{2\pi} = (1,0)$. On the other hand, if the circle is parameterized as $x = \sin t$, $y = \cos t$, then its orientation is clockwise, and $P_0 = P_{2\pi} = (0,1)$.

C is said to be a **simple curve** if it never crosses itself between $t = a$ and $t = b$. In other words, if t_1 and t_2 are two distinct values in the open interval (a,b), then $(x(t_1),y(t_1)) \neq (x(t_2),y(t_2))$.

When we stipulate that a curve is simple, this implies that the curve is traversed *exactly once* when t varies from a to b. In other words, we rule out a situation such as parameterizing the unit circle as $x = \cos t$, $y = \sin t$, with $t \in [0,4\pi]$, since here the circle would be traversed twice. However, if we stipulate $t \in [0,2\pi]$, then we have a simple, closed curve.

For any oriented simple, closed curve in the x,y plane:

- Its orientation can be classified as either *clockwise* or *counter-clockwise*.
- The curve determines a well-defined region of the x,y plane known as its **interior**, and the curve is the *boundary* of this region. (Its interior is always on your *left* when you traverse the curve *counter-clockwise*, and is always on your *right* when you traverse the curve *clockwise*.)

This is generally not the case for a simple, closed curve in x,y,z space. Furthermore, both in the x,y plane and in x,y,z space, it is generally not the case for a curve that is *not* closed or *not* simple.

Here is an intriguing example of a two-dimensional curve that is closed but not simple: The curve consisting of the line segment from $(1,0)$ to $(1,1)$, followed by the line segment from $(1,1)$ to $(0,1)$, followed by the line segment from $(0,1)$ to $(0,-2)$, followed by the line segment from $(0,-2)$ to $(-1,-2)$, followed by the line segment from $(-1,-2)$ to $(-1,2)$, followed by the line segment from $(-1,2)$ to $(2,2)$, followed by the line segment from $(2,2)$ to $(2,-2)$, followed by the line segment from $(2,-2)$ to $(-1,-2)$, followed by the line segment from $(-1,-2)$ to $(-1,0)$, followed by the line segment from $(-1,0)$ to $(1,0)$. This curve cannot be classified as clockwise or counter-clockwise, and its interior cannot be coherently defined.

If a curve is simple but not closed, its orientation can be specified by indicating which endpoint is the initial point and which endpoint is the final point.

If we picture a particle moving along the curve from its initial point to its final point, in accordance with the curve's parametric equations, then the particle has position vector $\mathbf{r}(t) = < x(t), y(t) >$, velocity vector $\mathbf{v}(t) = < x'(t), y'(t) >$, speed $v(t) = \sqrt{x'(t)^2 + y'(t)^2}$, unit tangent vector $\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = (x'(t)^2 + y'(t)^2)^{-1/2} < x'(t), y'(t) >$, acceleration vector $\mathbf{a}(t) = < x''(t), y''(t) >$, and arclength $s(t) = \int_a^t v(u)\, du$.

By the Fundamental Theorem of Calculus (Part One), $s'(t) = v(t)$.

C is said to be a **smooth curve** if it has no kinks (or cusps). This occurs provided $v(t)$ is never zero.

We shall deal with curves that are both simple and smooth. They may or may not be closed.

In Leibnitz notation, $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$, and $s'(t) = \frac{ds}{dt}$. It follows that $dx = x'(t)\, dt$, $dy = y'(t)\, dt$, and $ds = s'(t)\, dt$. Writing $v(t)$ in place of $s'(t)$, we have $ds = v(t)\, dt$.

Let $f(x,y)$ be a real-valued function whose domain includes the curve C. For this function and this curve, we can define three different definite integrals. They are called **line integrals**, but it would be more accurate to call them **curve integrals**, because we are integrating the function f over the curve C. Furthermore, these integrals are classified as **scalar line integrals** because the function f is real-valued. (In a little while, we will also study *vector field line integrals*.)

- $\int_C f(x,y)\, dx$ is called the line integral with respect to x.
- $\int_C f(x,y)\, dy$ is called the line integral with respect to y.
- $\int_C f(x,y)\, ds$ is called the line integral with respect to s (i.e., with respect to arc length).

Each integral is evaluated by expressing its integrand in terms of the parameter t.

- $\int_C f(x,y)\, dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt$
- $\int_C f(x,y)\, dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt$
- $\int_C f(x,y)\, ds = \int_a^b f(x(t),y(t))\, v(t)\, dt = \int_a^b f(x(t),y(t))\, \sqrt{x'(t)^2 + y'(t)^2}\, dt$

For the sake of brevity, we may write $f(t)$ in place of $f(x(t),y(t))$. Thus, we have:

- $\int_C f(x,y)\, dx = \int_a^b f(t)\, x'(t)\, dt$
- $\int_C f(x,y)\, dy = \int_a^b f(t)\, y'(t)\, dt$
- $\int_C f(x,y)\, ds = \int_a^b f(t)\, v(t)\, dt = \int_a^b f(t)\, \sqrt{x'(t)^2 + y'(t)^2}\, dt$

Of these three scalar line integrals, the third is the one that has the most straightforward application...

Preliminary Discussion:

In basic geometry, a "cylinder" is understood to be a circular cylinder with finite height, capped off by a circular top and a circular bottom. The cylinder has a *lateral face* that may be referred to as a *tube*. If the cylinder has radius r and height h, then the area of the tube is $2\pi rh$. To justify this formula, imagine slitting the tube vertically and then rolling it out flat, in which case it becomes a rectangle. The width of the rectangle is the height h. The length of the rectangle is the circumference of the cylinder's circular top and bottom, which is $2\pi r$. Hence, the rectangle's area is the product of its length and its width, giving us $2\pi rh$.

In calculus, a circular cylinder is the orthogonal projection of a circle into the third dimension, and it has infinite extension. Consider the circle $x^2 + y^2 = 9$ in the x,y plane, centered at the origin and having radius 3. Its orthogonal projection is the cylinder $x^2 + y^2 = 9$ in x,y,z space, which is parallel to the z axis. It extends infinitely far in the direction of both the positive z axis and the negative z axis. However, the section of the cylinder between the horizontal planes $z = 0$ and $z = 5$ is of finite extension; its height is 5. Using our basic geometry formula, the area of this section is $2\pi(3)(5) = 30\pi$.

Now suppose we change the upper boundary from the horizontal plane $z = 5$ to the oblique plane $x + z = 10$. This plane is the orthogonal projection, parallel to the y axis, of the line

$z = -x + 10$ in the x,z plane. We now have a section of our circular cylinder with a *flat bottom edge* and an *irregular top edge* (in other words, the bottom edge has a fixed height of 0, but the top edge has a variable height).

If we roll the section out flat, as we did before, we no longer have a rectangle. We have a two-dimensional shape with one horizontal straight edge, two vertical straight edges, and one curving edge. How do we find the area of this shape? Well, isn't that exactly why God created the definite integral?

Actually, we can find the area *without* rolling the section out flat. This is why God created the scalar line integral with respect to arc length!

The curve C is the circle $x^2 + y^2 = 9$ in the x,y plane. It may be parameterized as $x = 3\cos t$, $y = 3\sin t$, where $t \in [0, 2\pi]$. Here we have $f(x,y) = -x + 10 = -3\cos t + 10 = f(t)$. So the area is $\int_C (-x+10)ds = \int_0^{2\pi} (-3\cos t + 10)\ v(t)\ dt$. Since $x'(t) = -3\sin t$ and $y'(t) = 3\cos t$, $v(t) = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$, so we get
$3\int_0^{2\pi} (-3\cos t + 10)dt = 3[-3\sin t + 10t]_0^{2\pi} = 3(20\pi) = 60\pi.$

We can also have scalar line integrals in three dimensions. For these, we may write $f(t)$ in place of $f(x(t), y(t), z(t))$.

- $\int_C f(x,y,z)\ dx = \int_a^b f(x(t),y(t),z(t))\ x'(t)\ dt = \int_a^b f(t)\ x'(t)\ dt$
- $\int_C f(x,y,z)\ dy = \int_a^b f(x(t),y(t),z(t))\ y'(t)\ dt = \int_a^b f(t)\ y'(t)\ dt$
- $\int_C f(x,y,z)\ dz = \int_a^b f(x(t),y(t),z(t))\ z'(t)\ dt = \int_a^b f(t)\ z'(t)\ dt$
- $\int_C f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\ v(t)\ dt = \int_a^b f(x(t),y(t),z(t))\ \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt =$
  $\int_a^b f(t)\ v(t)\ dt = \int_a^b f(t)\ \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt$

Vector Field Line Integrals in Two Dimensions:

Let $\mathbf{F}(x,y) = \langle\ p(x,y),\ q(x,y)\ \rangle$ be a two-dimensional vector field, which we shall interpret as a force field. Assume the component functions p and q are continuous.

Let C be a curve in the x,y plane as outlined in the preceding section, with parametric equations $x = x(t)$, $y = y(t)$, where $t \in [a,b]$.

We now address the following physics issue: What is is the work done in moving a particle through the force field along the curve C from the point P_a to the point P_b (in other words, what is the work done during the time interval from $t = a$ to $t = b$)?

At any instant of time, the particle is located at the point $(x(t),y(t))$, and the force acting on the particle at this instant is $\mathbf{F}(x(t),y(t)) \ = < p(x(t),y(t)),\ q(x(t),y(t)) \ >$. We may denote this more compactly by writing $\mathbf{F}(t) = < p(t), q(t) >$.

For brevity, may write $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{T}$ in place of $\mathbf{r}(t)$, $\mathbf{v}(t)$, and $\mathbf{T}(t)$, and we may write v and s in place of $v(t)$ and $s(t)$. We may write $\mathbf{F}$ in place of either $\mathbf{F}(x,y)$ or $\mathbf{F}(t)$, we may write p in place of either $p(x,y)$ or $p(t)$, and we may write q in place of either $q(x,y)$ or $q(t)$. So $\mathbf{F} = < p,q >$.

It's possible that the force $\mathbf{F}$ might be acting in the same direction as $\mathbf{T}$, but typically it will not be–it will be acting in some other direction. In this case, the work done will depend on the *component of force acting in the direction of* $\mathbf{T}$, i.e., on the component of $\mathbf{F}$ along $\mathbf{T}$, $comp_{\mathbf{T}}\mathbf{F}$.

Here is a quick refresher on the subject...

Given two nonzero vectors $\mathbf{u}$ and $\mathbf{w}$, the **vector projection** of $\mathbf{w}$ **onto** $\mathbf{u}$ is denoted $proj_{\mathbf{u}}\mathbf{w}$. The **component** of $\mathbf{w}$ **along** $\mathbf{u}$, also known as the **scalar projection** of $\mathbf{w}$ **onto** $\mathbf{u}$, is denoted $comp_{\mathbf{u}}\mathbf{w}$.

$comp_{\mathbf{u}}\mathbf{w}$ is the **signed magnitude** of $proj_{\mathbf{u}}\mathbf{w}$. If the angle between $\mathbf{u}$ and $\mathbf{w}$ is less than or equal to $\frac{\pi}{2}$, then $comp_{\mathbf{u}}\mathbf{w} = |proj_{\mathbf{u}}\mathbf{w}|$, but if the angle is greater than $\frac{\pi}{2}$, then $comp_{\mathbf{u}}\mathbf{w} = -|proj_{\mathbf{u}}\mathbf{w}|$.

$comp_{\mathbf{u}}\mathbf{w} = \frac{\mathbf{u} \cdot \mathbf{w}}{u}$, where u is the magnitude of $\mathbf{u}$.

$$proj_{\mathbf{u}}\mathbf{w} = (comp_{\mathbf{u}}\mathbf{w})\frac{\mathbf{u}}{u} = \frac{\mathbf{u} \cdot \mathbf{w}}{u}\frac{\mathbf{u}}{u} = \frac{\mathbf{u} \cdot \mathbf{w}}{u^2}\mathbf{u}$$

If $\mathbf{u}$ is a unit vector, then $u = 1$, and our formulas simplify as follows:

$$comp_{\mathbf{u}}\mathbf{w} = \mathbf{u} \cdot \mathbf{w}$$

$$proj_{\mathbf{u}}\mathbf{w} = (comp_{\mathbf{u}}\mathbf{w})\mathbf{u} = (\mathbf{u} \cdot \mathbf{w})\mathbf{u}$$

Now let's get back to Chapter 16.

Since $\mathbf{T}$ is a unit vector, $comp_{\mathbf{T}}\mathbf{F} = \mathbf{T} \cdot \mathbf{F}$.

We normally write $\mathbf{F} \cdot \mathbf{T}$ in place of $\mathbf{T} \cdot \mathbf{F}$. (We can do so because the dot product is commutative.)

To find the work done by the force field, we integrate $\mathbf{F} \cdot \mathbf{T}$ over the curve C from the point P_a to the point P_b. To be more specific, we integrate with respect to arc length. Thus, we calculate the work done by evaluating the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$. We may call this the **work integral**. It is an example of a **vector field line integral**.

Substituting $\frac{1}{v}\mathbf{v}$ in place of $\mathbf{T}$ and $v\, dt$ in place of ds, $\mathbf{F} \cdot \mathbf{T}\, ds = \mathbf{F} \cdot (\frac{1}{v}\mathbf{v})\, v\, dt =$ $\frac{1}{v}(\mathbf{F} \cdot \mathbf{v})\, v\, dt = \mathbf{F} \cdot \mathbf{v}\, dt$. Thus, we may write our work integral as $\int_C \mathbf{F} \cdot \mathbf{v}\, dt$.

Since dt is a scalar, we may write $\mathbf{F} \cdot \mathbf{v}\, dt$ as $\mathbf{F} \cdot (\mathbf{v}\, dt)$. (If we were going to be really picky, we should write $\mathbf{v}\, dt$ as $dt\, \mathbf{v}$, since when we have a scalar multiple of a vector, we are supposed to write the scalar to the left of the vector, but let's not worry about this.)

Since $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, we have $d\mathbf{r} = \mathbf{v}\, dt$. So we may write $\mathbf{F} \cdot (\mathbf{v}\, dt)$ as $\mathbf{F} \cdot d\mathbf{r}$. Thus, we may write our work integral as $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Since $\mathbf{v} = < \frac{dx}{dt}, \frac{dy}{dt} >$, $\mathbf{F} \cdot \mathbf{v} = < p, q > \cdot < \frac{dx}{dt}, \frac{dy}{dt} > = p\frac{dx}{dt} + q\frac{dy}{dt}$. Consequently, $\mathbf{F} \cdot \mathbf{v}\, dt = \left(p\frac{dx}{dt} + q\frac{dy}{dt}\right)dt = p\, dx + q\, dy$. We may therefore write our work integral as $\int_C p\, dx + q\, dy$. This may also be written as $\int_C p\, dx + \int_C q\, dy$.

Notice that $\int_C p\, dx$ and $\int_C q\, dy$ are scalar line integrals with respect to x and y, respectively (as defined earlier).

In summary, our work integral can be expressed in any of the following ways (i.e., the following expressions are equivalent):

1. $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$
2. $\int_C \mathbf{F} \cdot \mathbf{v}\, dt$
3. $\int_C \mathbf{F} \cdot d\mathbf{r}$
4. $\int_C p\, dx + q\, dy = \int_C p\, dx + \int_C q\, dy$

As already discussed, we evaluate a line integral by rewriting it in terms of t. In version #2, we simply change the boundaries of integration to a and b. In version #4, we must also replace dx with $x'(t)\ dt$, and replace dy with $y'(t)\ dt$. Thus we get:

- $\displaystyle\int_a^b \mathbf{F}(t) \cdot \mathbf{v}(t)\ dt$
- $\displaystyle\int_a^b p(t)\ x'(t)\ dt + q(t)\ y'(t)\ dt = \int_a^b \left[p(t)\ x'(t) + q(t)\ y'(t)\right] dt$, or

  $\displaystyle\int_a^b p(t)\ x'(t)\ dt + \int_a^b q(t)\ y'(t)\ dt$

Example 1:

Let $\mathbf{F}(x,y) = < y - x,\ x >$, and let C be the quarter of the unit circle from the point $(0,1)$ to the point $(1,0)$, parameterized as $x = \sin t,\ y = \cos t$, where $t \in [0, \frac{\pi}{2}]$. (Note that the positive direction of the curve is clockwise.) Find the work done in moving a particle through the force field along curve C from $(0,1)$ to $(1,0)$.

Here, $p(x,y) = y - x$ and $q(x,y) = x$, so $p(t) = \cos t - \sin t$ and $q(t) = \sin t$.

$x'(t) = \cos t$ and $y'(t) = -\sin t$.

$p(t)\ x'(t) = (\cos t - \sin t)\cos t = \cos^2 t - \cos t \sin t$.

$q(t)\ y'(t) = \sin t(-\sin t) = -\sin^2 t$.

The work is thus $\displaystyle\int_0^{\pi/2} [\cos^2 t - \cos t \sin t - \sin^2 t]\ dt$.

We may employ the trig identities $\cos^2 t - \sin^2 t = \cos 2t$, and $\cos t \sin t = \frac{1}{2}\sin 2t$.

$$\int_0^{\pi/2} (\cos^2 t - \sin^2 t - \cos t \sin t)\ dt = \int_0^{\pi/2} \left(\cos 2t - \tfrac{1}{2}\sin 2t\right) dt =$$

$$\left[\tfrac{1}{2}\sin 2t + \tfrac{1}{4}\cos 2t\right]_0^{\pi/2} = -\tfrac{1}{2}.$$

Notice that work can be negative as well as positive (or, indeed, zero). The *sign of the work* depends on the *orientation of the curve*. Reversing the orientation of the curve reverses the sign of the work.

If C denotes an oriented curve, then the curve consisting of the same points but with the reverse orientation may be denoted $-C$. So $\int_{-C} \mathbf{F} \cdot \mathbf{T}\, ds = -\int_{C} \mathbf{F} \cdot \mathbf{T}\, ds$

In the above example, $-C$ would be the quarter of the unit circle starting at $(1,0)$ and ending at $(0,1)$, with a counter-clockwise orientation, parameterized as $x = \cos t,\ y = \sin t,$ where $t \in [0, \frac{\pi}{2}]$. The work done in moving a particle through $\mathbf{F}$ along *this* curve would be $\frac{1}{2}$.

A given curve can have many different parameterizations all with the *same* orientation. The work done in moving a particle along the curve is the same *regardless* of which of these parameterizations is used. In other words, the work integral is *independent of parameterization*, so long as orientation is preserved.

So far, we have been dealing with curves that are simple and smooth. However, we can also deal with line integrals over curves that are simple and **piecewise smooth**. Say we have a simple, smooth curve C_1 with initial point P_1 and final point Q_1, and another simple, smooth curve C_2 with initial point P_2 and final point Q_2. Suppose $P_2 = Q_1$, i.e., the second curve starts where the first curve ends. Suppose the curves are otherwise disjoint, with the possible exception that Q_2 might equal P_1. The curve $C = C_1 \cup C_2$ is considered to be simple. It may or may not be smooth, depending on whether it has a kink at Q_1, but in either case it is considered to be *piecewise smooth* (because it is the union of two smooth "pieces" or subcurves).

A simple, piecewise smooth curve is also known as a **path**.

Given the piecewise smooth curve $C = C_1 \cup C_2$, $\int_{C} = \int_{C_1} + \int_{C_2}$ (this applies to all of the different types of line integral discussed so far).

The curve $C = C_1 \cup C_2$, as outlined above, has initial point P_1 and final point Q_2. We could think of a particle moving along the curve from P_1 to Q_2, passing through the point Q_1 along the way. Suppose it takes 3 seconds for it to journey from P_1 to Q_1, and another 3 seconds for it to journey from Q_1 to Q_2 (so the entire journey takes 6 seconds). We could parameterize C so the journey from P_1 to Q_2 occurs when t varies over the time interval $[0,6]$. However, the work done by the force field $\mathbf{F}$ in this case would be equal to the work done if we had *two* particles, one moving from P_1 to Q_1 and the other *simultaneously* moving from Q_1 to Q_2. We can parameterize C_1 and C_2 so that both journeys occur over the time interval $[0,3]$. This is generally the model we will adopt.

Before we consider an example, here is a reminder of an important principle from Module One: Say we want to parameterize a line segment with endpoints A and B, so that A is the initial point and B is the final point, with $t \in [0,1]$.

- If the line segment is two-dimensional, let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Then we may use parametric equations $x = (1-t)a_1 + tb_1,\ y = (1-t)a_2 + tb_2$.

- If the line segment is three-dimensional, let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then we may use parametric equations $x = (1-t)a_1 + tb_1$, $y = (1-t)a_2 + tb_2$, $z = (1-t)a_3 + tb_3$.

Example 2:

In the x,y plane, let $C = C_1 \cup C_2$, where C_1 is the line segment with endpoints $(1,0)$ and $(0,1)$, while C_2 is the line segment with endpoints $(0,1)$ and $(-1,0)$. We shall set up parameterizations so that C_1 has initial point $(1,0)$ and final point $(0,1)$, and C_2 has initial point $(0,1)$ and final point $(-1,0)$. We can think of C itself as having initial point $(1,0)$ and final point $(-1,0)$.

We parameterize C_1 by the parametric equations $x_1(t) = 1-t$, $y_1(t) = t$, with $t \in [0,1]$. Note that $x_1{}'(t) = -1$ and $y_1{}'(t) = 1$.

We parameterize C_2 by the parametric equations $x_2(t) = -t$, $y_2(t) = 1-t$, with $t \in [0,1]$. Note that $x_2{}'(t) = -1$ and $y_2{}'(t) = -1$.

Let $\mathbf{F}(x,y) = < x+y, x-y >$. Here we have $p(x,y) = x+y$ and $q(x,y) = x-y$.

On C_1, $\mathbf{F}(x_1(t), y_1(t)) \; = < x_1(t) + y_1(t), x_1(t) - y_1(t) > = < 1, 1-2t >$. Here we have $p_1(t) = 1$ and $q_1(t) = 1-2t$.

On C_2, $\mathbf{F}(x_2(t), y_2(t)) \; = < x_2(t) + y_2(t), x_2(t) - y_2(t) > = < 1-2t, -1 >$. Here we have $p_2(t) = 1-2t$ and $q_2(t) = -1$.

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds =$$

$$\int_0^1 \left[p_1(t)\, x_1{}'(t) + q_1(t)\, y_1{}'(t) \right] dt + \int_0^1 \left[p_2(t)\, x_2{}'(t) + q_2(t)\, y_2{}'(t) \right] dt =$$

$$\int_0^1 [(1)(-1) + (1-2t)(1)]\, dt + \int_0^1 [(1-2t)(-1) + (-1)(-1)]\, dt =$$

$$\int_0^1 -2t\, dt + \int_0^1 2t\, dt = [-t^2]_0^1 + [t^2]_0^1 = -1 + 1 = 0.$$

Example 3:

In the x,y plane, let C be the semicircle $x^2 + y^2 = 1$ with $y \geq 0$, having endpoints $(1,0)$ and $(-1,0)$, parameterized as $x = \cos t$, $y = \sin t$, where $t \in [0,\pi]$. $x'(t) = -\sin t$ and $y'(t) = \cos t$. Thus, $(1,0)$ is the initial point and $(-1,0)$ is the final point.

As in Example 2, let $\mathbf{F}(x,y) = < x+y, x-y >$. Here, $p(x,y) = x+y$ and $q(x,y) = x-y$, so $p(t) = \cos t + \sin t$ and $q(t) = \cos t - \sin t$.

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^\pi [(\cos t + \sin t)(-\sin t) + (\cos t - \sin t)(\cos t)]\, dt =$$

$$\int_0^\pi [\cos^2 t - \sin^2 t - 2\cos t \sin t]\, dt = \int_0^\pi [\cos 2t - \sin 2t]\, dt =$$

$$\left[\tfrac{1}{2}\sin 2t + \tfrac{1}{2}\cos 2t\right]_0^\pi = \tfrac{1}{2}[\sin 2t + \cos 2t]_0^\pi = \tfrac{1}{2}[(0-0) + (1-1)] = 0.$$

Notice that Examples 2 and 3 involve the same vector field $\mathbf{F}$, but two different curves C. Both curves have the same initial point, $(1,0)$, and the same final point $(-1,0)$. Since they are simple curves, we can therefore say they have the *same orientation*. Notice also that the work done in moving a particle along each curve is the same. This is not a coincidence. This vector field $\mathbf{F}$ is conservative, and a conservative vector field has a very important property known as **independence of path**. This means that if two paths in the vector field have the same initial point and the same final point, then the work integrals for the two paths will be equal. In other words, the work done in moving a particle from one point to another is the same regardless of the path by which the particle travels from the one point to the other.

For the vector field $\mathbf{F}(x,y) = < p(x,y),\ q(x,y) >$ and the curve C with parametric equations $x = x(t),\ y = y(t),$ where $t \in [a,b]$, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C p\, dx + q\, dy = \int_a^b p(t)\, x'(t)\, dt + q(t)\, y'(t)\, dt$ is called it a *work integral* because its most prominent application is calculating the work done in moving a particle through a force field from the initial point of C to the final point.

The vector field line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ has many other applications as well. In certain applications, $\mathbf{F}$ represents **fluid flow**, rather than a force field. In these applications, C is generally a simple, closed curve. In this context, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is known as a **circulation integral**. To indicate that the curve C is closed, we may write $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$. (The Stewart text stipulates that this notation requires C to have a counter-clockwise orientation, also known as a *positive orientation.* However, other texts, such as Briggs, do not require this.)

The work integral or circulation integral is formed by integrating, over the curve C, the component of $\mathbf{F}$ in the direction of $\mathbf{T}$, $comp_{\mathbf{T}}\mathbf{F}$, which is $\mathbf{F} \cdot \mathbf{T}$. In other words, we integrate the component of $\mathbf{F}$ that is *tangential* to the curve (in the direction of motion).

The work or circulation integral is only one kind of vector field line integral. Another kind is known as a **flux integral**. This arises in connection with fluid flow, and is formed by integrating, over the curve C, the component of $\mathbf{F}$ that is *normal* to the curve (in other words, normal or orthogonal or perpendicular to $\mathbf{T}$). This will be $comp_{\mathbf{n}}\mathbf{F}$, which is $\mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ is an appropriately selected *unit normal vector* (a unit vector normal to the tangent line).

Earlier this semester, we defined a curve's *principal unit normal vector* as $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, which is orthogonal to $\mathbf{T}(t)$ and points in the direction in which the curve is turning (known as the *direction of curvature*). However, for present purposes, we must consider a different unit normal vector, which we will call the *flux unit normal vector*.

Recall that $\mathbf{T}(t)$ was defined as $\frac{1}{v(t)}\mathbf{v}(t) = (x'(t)^2 + y'(t)^2)^{-1/2} < x'(t), y'(t) >$ (in other words, $\mathbf{T}$ is the unit vector in the direction of $\mathbf{v}$). Since $\mathbf{v} = < x'(t), y'(t) >$, the vector $< y'(t), -x'(t) >$ is orthogonal to $\mathbf{v}$, and therefore also orthogonal to $\mathbf{T}$. $< y'(t), -x'(t) >$ has the same magnitude as $\mathbf{v}$ (i.e., its magnitude is v), so $\frac{1}{v} < y'(t), -x'(t) >$ is a *unit* vector orthogonal to $\mathbf{T}$, and hence normal to the tangent line–i.e., it is a *unit normal vector*. We shall define this as our **flux unit normal vector**, which we will denote as $\mathbf{n}(t)$. Thus, $\mathbf{n}(t) = (x'(t)^2 + y'(t)^2)^{-1/2} < y'(t), -x'(t) >$. More briefly, $\mathbf{n} = \frac{1}{v} < y', -x' >$. For any value of t, if $\mathbf{n}(t)$ and $\mathbf{T}(t)$ are positioned with a common tail, typically the point $(x(t), y(t))$ on curve C, then $\mathbf{n}(t)$ points to the *right* of $\mathbf{T}(t)$.

Recall that a simple, closed curve in the x,y plane has a well-defined interior, and if the curve has a counter-clockwise orientation, then its interior always lies to the *left* of $\mathbf{T}$ (which implies that the *exterior* always lies to the *right*). Assuming this orientation, $\mathbf{n}$ always points to the curve's exterior (and $-\mathbf{n}$ always points to the curve's interior). Thus, we may refer to $\mathbf{n}$ as the **outward unit normal vector**, and we may refer to $-\mathbf{n}$ as the **inward unit normal vector**. For example, if C is a circle centered at the origin, then $-\mathbf{n}$ always points toward the origin, while $\mathbf{n}$ always points away from the origin (assuming the circle is oriented counter-clockwise).

For the sake of brevity, the flux unit normal vector, $\mathbf{n} = \frac{1}{v} < y', -x' >$, may be referred to simply as the *flux vector*. If C is a simple, closed curve with a counter-clockwise orientation, we may refer to $\mathbf{n}$ as the **outward flux vector**.

The vector field line integral $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$ is known as a **flux integral**. If C is a closed curve oriented counter-clockwise, then we write this as $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$, and we may call it the **outward flux integral**.

It can be shown that $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C p\, dy - q\, dx = \int_a^b [p(t)\, y'(t) - q(t)\, x'(t)]\, dt$

Work and Circulation Integrals in Three Dimensions:

Let $\mathbf{F}(x,y,z) = < p(x,y,z),\ q(x,y,z),\ r(x,y,z) >$ be a three-dimensional vector field, where the component functions $p,\ q,$ and r are continuous.

Let C be a simple, piecewise-smooth curve in x,y,z space with parametric equations $x = x(t),\ y = y(t),\ z = z(t),$ where $t \in [a,b]$. The initial point of C is $P_a = (x(a),y(a),z(a))$ and the final point is $P_b = (x(b),y(b),z(b))$.

We may write $\mathbf{F}(t) = < p(t),q(t),r(t) >$ in place of
$\mathbf{F}(x(t),y(t),z(t)) \ = < p(x(t),y(t),z(t)),\ q(x(t),y(t),z(t)),\ r(x(t),y(t),z(t)) \ >.$

The work or circulation integral can be expressed as:

1. $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\ ds$
2. $\displaystyle\int_C \mathbf{F} \cdot \mathbf{v}\ dt$
3. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$
4. $\displaystyle\int_C p\ dx + q\ dy + r\ dz = \int_C p\ dx + \int_C q\ dy + \int_C r\ dz$

We evaluate by rewriting the integral in terms of t.

- $\displaystyle\int_a^b \mathbf{F}(t) \cdot \mathbf{v}(t)\ dt$
- $\displaystyle\int_a^b p(t)\ x'(t)\ dt + q(t)\ y'(t)\ dt + r(t)\ z'(t)\ dt = \int_a^b \left[p(t)\ x'(t) + q(t)\ y'(t) + r(t)\ z'(t)\right] dt,$ or
  $\displaystyle\int_a^b p(t)\ x'(t)\ dt + \int_a^b q(t)\ y'(t)\ dt + \int_a^b r(t)\ z'(t)\ dt$

Example 4:

Let $\mathbf{F}(x,y,z) = < z,x,-y >$. Let C be the tilted ellipse $x = \cos t,\ y = \sin t,\ z = \cos t,$ where $t \in [0,2\pi]$. Find $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\ ds.$

Here, $p(x,y,z) = z,\ \ q(x,y,z) = x,$ and $r(x,y,z) = -y$. Thus:

- $p(t) = \cos t,\ \ q(t) = \cos t,$ and $r(t) = -\sin t.$
- $x'(t) = -\sin t,\ \ y'(t) = \cos t,$ and $z'(t) = -\sin t.$

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_0^{2\pi} [(\cos t)(-\sin t) + (\cos t)(\cos t) + (-\sin t)(-\sin t)]\, dt =$$

$$\int_0^{2\pi} (-\cos t \sin t + \cos^2 t + \sin^2 t))dt = \int_0^{2\pi} (1 - \cos t \sin t)dt = \int_0^{2\pi} \left(1 - \tfrac{1}{2}\sin 2t\right)dt =$$

$$\left[t + \tfrac{1}{4}\cos 2t\right]_0^{2\pi} = (2\pi + \tfrac{1}{4}) - (0 + \tfrac{1}{4}) = 2\pi$$