3.9 Related Rates

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity. For example: As a spherical balloon is being inflated, the radius is increasing and the volume is also increasing. How does the rate of increase of the radius compare to the rate of increase of the volume?

STRATEGY:
1. Read the problem carefully until you understand what quantities are being given and what quantities you are asked to find.
2. Draw a diagram, if possible. (remember – a picture is worth a 1000 words!)
3. Introduce notation: Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives. (Implicit differentiation will be used!)
5. Write an equation that relates the various quantities of the problem. If necessary, use geometry of the situation to eliminate one of the variables by substitution.
6. Use the chain rule to differentiate both sides of the equation with respect to ***t***.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Let's start out with some easy problems.

Examples:

a) If $y = x^3 + 2x$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 2$. Take the derivative of the function with respect to $\boldsymbol{t}$. This will require implicit differentiation. I will use Leibniz notation since we are taking the derivative with respect to $\boldsymbol{t}$.

$\frac{d}{dt}[y = x^3 + 2x] \quad \Rightarrow \quad \frac{dy}{dt} = 3x^2\frac{dx}{dt} + 2\frac{dx}{dt}$ Now use $x = 2$ and $\frac{dx}{dt} = 5$ to solve for $\frac{dy}{dt}$

$\frac{dy}{dt} = 3(2)^2 \cdot 5 + 2 \cdot 5 \Rightarrow \frac{dy}{dt} = 3 \cdot 4 \cdot 5 + 2 \cdot 5 \quad \Rightarrow \quad \boldsymbol{\frac{dy}{dt} = 70}$

b) If $z^2 = x^2 + y^2$, $\frac{dx}{dt} = 2$ *and* $\frac{dy}{dt} = 3$, *find* $\frac{dz}{dt}$ *when* $x = 5$ *and* $y = 12$.

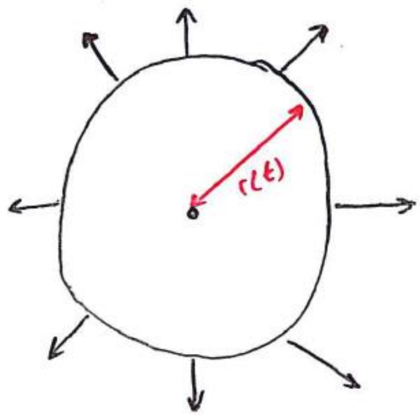
Take the derivative of the function with respect to $\boldsymbol{t}$. $\frac{d}{dt}[z^2 = x^2 + y^2] \quad \Rightarrow \quad 2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

Notice that we were given **x** and **y** but we don't have a value for **z**. But using the Pythagorean Thm., we can find **z.** $\mathbf{z^2 = 5^2 + 12^2}$ $\quad \mathbf{z = \sqrt{25 + 144}} \quad \mathbf{z = \sqrt{169}} \quad \mathbf{z = \pm 13.}$ Now use the given information with the new information to find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}}{2z} \qquad \frac{dz}{dt} = \pm\frac{2 \cdot 5 \cdot 2 + 2 \cdot 12 \cdot 3}{2 \cdot 13} \qquad \frac{dz}{dt} = \pm\frac{92}{26} \qquad \boldsymbol{\frac{dz}{dt} = \pm\frac{46}{13}}$$

Now let's attack some word problems. (Applications)

Example: An oil rig springs a leak in calm seas, and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of **30m/hr**, how fast is the area of the patch increasing when the patch has a radius of **100 meters**? Draw a picture.



Notice that the radius and area change simultaneously.

The relationship between the radius and the area is the formula:

$$A = \pi r^2$$

Also, at a certain time, t, the radius is ***r(t)***, and the area is ***A(t).***

Therefore the general equation relating the radius and area at time t is

$$\boldsymbol{A(t) = \pi(r(t))^2}$$

The question is asking to find the rate of change of the area, $\frac{dA}{dt}$, (that can also be denoted as ***A'(t)***), given that the rate of change of the radius, $\frac{dr}{dt}$, or ***r'(t)*** = 30 m/hr.

To find ***A'(t)***, we need to differentiate ***A(t)*** = $\boldsymbol{\pi(r(t))^2}$ with respect to ***t***.

$$\boldsymbol{A'(t) = \frac{d}{dt}[\pi(r(t))^2]}$$

$$\boldsymbol{= \pi \frac{d}{dt}[(r(t))^2]}$$

$$\boldsymbol{= \pi \cdot 2 \cdot r(t) \cdot r'(t)} \quad \textbf{(chain rule)}$$

$$\boldsymbol{A'(t) = 2\pi r(t) r'(t)}$$

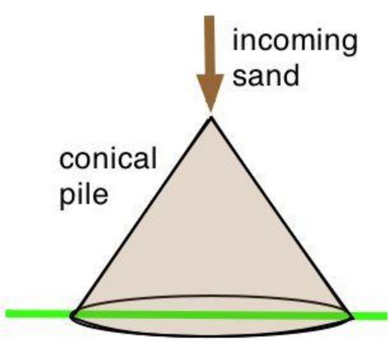
Now substitute ***r(t) = 100*** m, ***r'(t) = 30*** m/hr

$$\boldsymbol{A'(t) = 2\pi(100m)\left(\frac{30m}{hr}\right)}$$

$$\boldsymbol{A'(t) = 6000\pi \; m^2/hr}$$

This means that the area that the oil spill covers is increasing at a rate of 6000 $\boldsymbol{\pi \frac{m^2}{hr}}$ **or approximately 18,849.6 square meters per hour.**

Example: Sand falls from an overhead bin, accumulating in a conical pile with a radius that is always three time the height. If the sand falls from the bin at a rate of **120 ft³/min**, how fast is the height of the sand pile changing when the pile is **10 feet** high?



According to the problem **r = 3h**. The relationship of the variables is the Volume a cone given as $\boldsymbol{V = \frac{1}{3}\pi r^2 h}$ but we can substitute **3h** in for **r** making the formula $\boldsymbol{V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3}$. Since the volume and height are increasing with respect to time, t, we denote them as functions ***h(t)*** and ***V(t)***. So our formula looks like this:

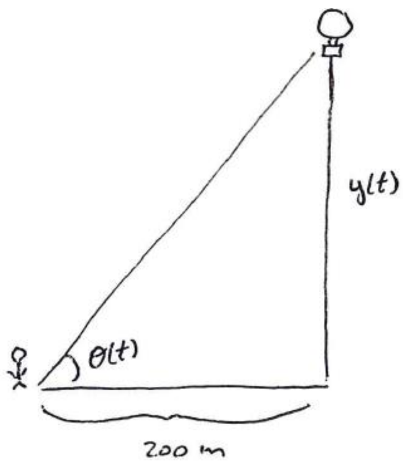
$\boldsymbol{V(t) = 3\pi(h(t))^3}$ We want to find the rate at which the height is changing as the volume changes so we need to take the derivative of the volume formula with respect to t. $\frac{d}{dt}[\boldsymbol{V(t) = 3\pi(h(t))^3}] =$

$\frac{dV}{dt} = 3\pi \frac{d}{dt}[(\boldsymbol{h(t)})^3] \Rightarrow 3\pi \cdot 3h(t)^2 \frac{dh}{dt}$ Solve for $\frac{dh}{dt}$. $\frac{dh}{dt} = \frac{\frac{dV}{dt}}{9\pi h(t)^2}$

Substitute the rate of change of the volume $\left(\frac{dV}{dt} = 120ft^3/min\right)$ and the height $(h(t) = 10\ ft)$ into the formula. $\frac{dh}{dt} = \frac{120\ ft^3}{9\pi(10ft)^2} = \frac{120\ ft^3}{900\pi ft^2} = \mathbf{0.042\ ft/min}$. **This means that when the sand pile is 10 feet high, the height is changing at a rate of 0.042 feet per minute.**

Example: An observer stands 200 meters from the launch site of a hot air balloon. The balloon rises vertically at a constant rate of 4 meters/second. How fast is the angle of elevation of the balloon increasing 30 seconds after the launch?

Note: The angle of elevation is the angle between the ground and the observer's line of sight of the balloon.



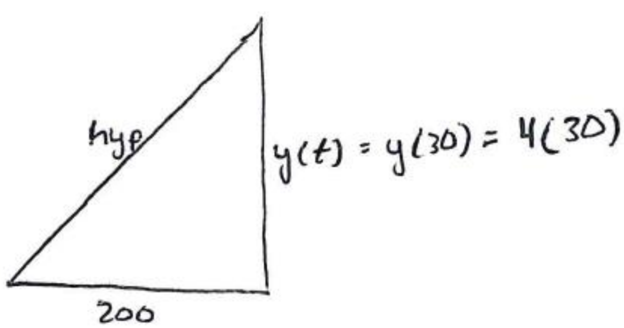
Notice that the angle of elevation, θ, changes as $\boldsymbol{y}$ changes. At a certain time, $\boldsymbol{t}$, the height of the balloon is ***y(t)*** and the angle of elevation is ***θ(t).*** The relationship between $\boldsymbol{\theta}$ and $\boldsymbol{y}$ is $\boldsymbol{tan(\theta)} = \frac{\boldsymbol{y}}{\mathbf{200}}$. At a certain time, $\boldsymbol{t}$ is given by $\boldsymbol{tan(\theta(t))} = \frac{\boldsymbol{y(t)}}{\mathbf{200}}$.

We need to find $\frac{d\theta}{dt}$, so differentiate $\boldsymbol{tan(\theta(t))} = \frac{\boldsymbol{y(t)}}{\mathbf{200}}$ with respect to $\boldsymbol{t}$.

$\frac{d}{dt}\left[\boldsymbol{tan(\theta(t))} = \frac{\boldsymbol{y(t)}}{\mathbf{200}}\right] \Rightarrow \frac{d}{dt}[\boldsymbol{tan(\theta(t))}] = \frac{d}{dt}\left[\frac{\boldsymbol{y(t)}}{\mathbf{200}}\right] \Rightarrow sec^2(\theta(t))\frac{d\theta}{dt} = \frac{1}{200}\frac{dy}{dt}$ Now solve for $\frac{d\theta}{dt}$.

$$\frac{d\theta}{dt} = \frac{1}{200sec^2(\theta(t))}\frac{dy}{dt} = \frac{cos^2(\theta(t))}{200}\frac{dy}{dt}$$

Notice that we don't know the value of $\cos(\theta(t))$. Remember that $\cos\theta(t) = \frac{adj.}{hyp.}$ *at* $t = 30$ *seconds*. We need the hypotenuse at ***t = 30.*** Since the height, ***y(t)***, is increasing at a rate of ***4 meters / second*** we can substitute and use trigonometry to find the hypotenuse.



The hyp $= \sqrt{120^2 + 200^2} \approx 233.24$ meters.

Therefore, $\cos\theta = \frac{200\ m}{233.24\ m} \approx 0.86$

This gives us that $\frac{d\theta}{dt} = \frac{0.86^2}{200\ m} \cdot 4\frac{m}{sec} = 0.015$ rad/sec.

This means that at t = 30 seconds, the balloon is rising at an angular rate of 0.015 radians per second.