10.2 Calculus with Parametric Curves

In this section we will use parametric equations to solve problems involving tangents, area, arc length and surface areas.

- **Tangents:**

Since parametric equations express a relationship between the variables **x** and **y**, it makes sense to ask about the derivative, $\frac{dy}{dx}$, at a certain point on the parametric curve.

If we know how to compute $\frac{dy}{dx}$, it can be used to determine slopes of lines tangent to the parametric curves.

Suppose **f** and **g** are differentiable functions and we want to find the tangent line at a point on the parametric curve **x = f(t)**, **y = g(t)**, where **y** is also a differentiable function of **x**. Then the chain rule gives:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we can solve for $\frac{dy}{dx}$. Using algebra, we can rewrite the above equation as

$$\boldsymbol{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad if \ \frac{dx}{dt} \neq 0}$$

In other words,

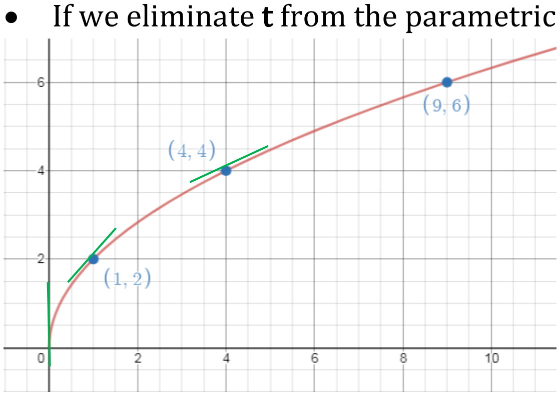
$$\boldsymbol{\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \quad if \ f'(t) \neq 0}$$

Example: Find $\frac{dy}{dx}$ for the following curves. Interpret the result and determine the points (if any) at which the curve has a horizontal or vertical tangent line.

$$a)\ x = f(t) = t, \quad y = g(t) = 2\sqrt{t}, \qquad for\ t\ \geq 0$$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{\frac{1}{\sqrt{t}}}{1} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{t}}}, \boldsymbol{provided\ t \neq 0}$$

- Notice that $\frac{dy}{dx}$ cannot equal 0 for **t > 0**. Therefore, the curve has no horizontal tangent lines.
- Also, as $\boldsymbol{t} \to \mathbf{0}^+, \boldsymbol{\frac{dy}{dx}} \to \infty$ which means that the curve has a vertical tangent line at (0, 0).
- If we eliminate **t** from the parametric equations we get $\boldsymbol{y = 2\sqrt{x}}$. See the graph below



- Now if we want to find the slopes of tangent lines at other points on the curve, we simply substitute the corresponding values of **t**. For example, the point (1, 2) corresponds to **t = 1** and the slope of the tangent line at that point is $\frac{1}{\sqrt{1}} = 1$.

$$b)\ x = f(t) = 4\cos(t), \quad y = g(t) = 16\sin(t) \quad for\ 0 \le t \le 2\pi$$

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = \frac{16\cos(t)}{-4\sin(t)} = -4\cot(t)$$

- At $t = 0, \pi, 2\pi, \cot(t)$ is undefined. Notice that

$$\lim_{t\to 0+} \frac{dy}{dx} = \lim_{t\to 0+} (-4\cot(t)) = -\infty \quad and \quad \lim_{t\to 0-} \frac{dy}{dx} = \lim_{t\to 0-} (-4\cot(t)) = \infty$$

Therefore, a vertical tangent line occurs at points corresponding to **t = 0, π.**

When **t = 0**

x = 4cos(0) = 4 → (4, 0)

y = 16sin(0) = 0 → (4, 0)

When **t = π**

x = 4cos(π) = -4 → (-4, 0)

y = 16sin(π) = 0 → (-4, 0)

- At $t = \frac{\pi}{2}, \frac{3\pi}{2}, \cot(t) = 0$. Therefore, a horizontal tangent line occurs at points corresponding to $t = \frac{\pi}{2}\ and\ \frac{3\pi}{2}$.

When $\boldsymbol{t} = \frac{\boldsymbol{\pi}}{\mathbf{2}}$

x = 4cos($\frac{\pi}{2}$) = 0 → (0, 16)

y = 16sin($\frac{\pi}{2}$) = 16 → (0, 16)

When $\boldsymbol{t} = \frac{\mathbf{3\pi}}{\mathbf{2}}$

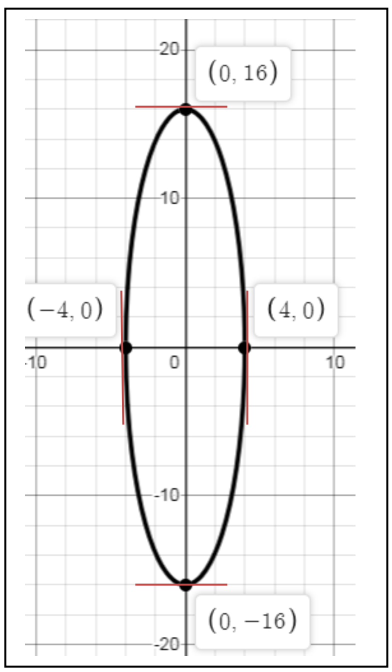
x = 4cos($\frac{3\pi}{2}$) = 0 → (0, -16)

y = 16sin($\frac{3\pi}{2}$) = -16 → (0, -16)

- Just like the last example, slopes of tangent lines at other points on the curve are found by substituting their corresponding values of **t.**
- Eliminating the variable **t**, we get:

$$\frac{x^2}{16} + \frac{y^2}{256} = 1$$

... which is an ellipse. See the graph at the right.



- **Areas**

We know that the area under the curve **y = f(x)** from [a, b] is $A = \int_a^b f(x)dx$, where $f(x) \geq 0$. If the curve is traced out once by the parametric equations **x = f(t)** and **y = g(t),** $\boldsymbol{\alpha \leq t \leq \beta}$, then we can calculate an area formula by using the substitution rule for definite integrals.

$$\boldsymbol{A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)dt \quad or \quad \left[\int_\beta^\alpha g(t)f'(t)dt\right]}$$

Example: Find the area under the curve when $x = f(t) = 4\cos(t), y = g(t) = 16\sin(t), for\ 0 < t < \pi$

Using the substitution rule: $g(t) = 16\sin(t), \quad f'(t) = -4\sin(t)$

$\frac{1-\cos(2t)}{2}$

$$A = \int_0^\pi 16\sin(t)\cdot -4\sin(t)dt = -64\int_0^\pi sin^2(t)dt = -\frac{64}{2}\int_0^\pi (1-\cos(2t))\,dt$$

$$= -32\left[t - \frac{1}{2}\sin(2t)\right]_0^\pi = -32\left[\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right] = \boldsymbol{-32\pi}$$

Since we know that the area is above the **x – axis**, therefore the area is **32π.** This is a case where we would reverse the **α** and **β**

$$A = -\int_\pi^0 16\sin(t)\cdot -4\sin(t)\,dt = -(-32)\int_\pi^0 1 - \cos(2t)dt \cdots \boldsymbol{32\pi}$$

- **Arc Length**

From previous sections, we have that the arc length **L** of a curve **C** from [a, b], assuming **y = f(x)** and **f '(x)** is continuous, is

$$\boldsymbol{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

If the curve **C** can be described by parametric equations **x = f(t)** and **y = g(t)**, **α < t < β**, where $\frac{dx}{dt} = f'(t) > 0$. Using the formula above, we obtain:

$$\boldsymbol{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt}$$

Since $\frac{dx}{dt} > 0$

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}} \frac{dx}{dt} dt = \int_{\alpha}^{\beta} \sqrt{\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}} \frac{dx}{dt} dt = \int_{\alpha}^{\beta} \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2}} \frac{dx}{dt} dt$$

$$= \int_{\alpha}^{\beta} \frac{1}{\frac{dx}{dt}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \frac{dx}{dt} dt = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If **x = f(t)** and **y = g(t)** then $\frac{dx}{dt} = f'(t)$ *and* $\frac{dy}{dt} = g'(t)$ then we can say the arc length is

$$L = \int_{\alpha}^{\beta} \sqrt{(f'(t))^2 + (g'(t))^2} \, dt$$

Theorem: If a curve **C** is described by the parametric equations **x = f(t)** and **y = g(t),** $\alpha \le t \le \beta$**,** where f' and g' are continuous on [a, b], and **C** is traversed exactly once as **t** increases from **α** to **β**, then the length of **C** is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Example: Find the exact length of the curve. $x = 1 + 3t^2, \; y = 4 + 2t^3, \; 0 \le t \le 1$

$$\frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = 6t^2$$

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2} dt = \int_0^1 \sqrt{36t^2 + 36t^4} dt = \int_0^1 \sqrt{36t^2(1+t^2)} \, dt = 6\int_0^1 \sqrt{1+t^2} \cdot t dt$$

Using **u – substitution**, let **u =** $1+t^2$ then **du** $= 2tdt \to \frac{1}{2}$**du** $= tdt$ when t = 0 → u = 1, when t = 1→ u = 2

$$L = 6\int_1^2 \sqrt{u} \, \frac{1}{2} du = 3\int_1^2 u^{\frac{1}{2}} du = 3\left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^2 = 3 \cdot \frac{2}{3}\left[2^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = 2\left[2\sqrt{2} - 1\right] = \mathbf{4\sqrt{2} - 2}$$

- **Surface Area**

The surface area equation is given by:

$$S = \int_{\alpha}^{\beta} 2\pi \cdot y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Assuming that **x** and **y** represent parametric equations and the curve is rotated about the x – axis.

Example: Find the exact area of the surface area generated by rotating the given curve about the x – axis.

$$x = t^3, \qquad y = t^2, \qquad 0 \le t \le 1$$

$$S = \int_0^1 2\pi \cdot t^2 \sqrt{(3t^2)^2 + (2t)^2}\,dt = 2\pi \int_0^1 t^2\sqrt{9t^4 + 4t^2}\,dt = 2\pi \int_0^1 t^2\sqrt{t^2(9t^2+4)}\,dt$$

$$= 2\pi \int_0^1 t^2 \cdot t\sqrt{9t^2+4}\,dt$$

(Let **u** $= 9t^2 + 4$ and $t^2 = \frac{u-4}{9}$ **du** $= 18tdt \to \frac{1}{18}\boldsymbol{du} = tdt$ when t = 0 → u = 4, when t = 1 → u = 13

$$= 2\pi \int_4^{13} \left(\frac{u-4}{9}\right)\sqrt{u} \cdot \frac{1}{18}du = \frac{2\pi}{18 \cdot 9}\int_4^{13} (u-4)\sqrt{u}\,du = \frac{\pi}{81}\int_4^{13}\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$$

$$= \frac{\pi}{81}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_4^{13} = \cdots (lots\ of\ algebra) = \boldsymbol{\frac{\pi}{1215}\left(494\sqrt{13} + 128\right)}$$

Example: Find the surface area generated by rotating the given curve about the **y – axis**.

$$x = e^t - t, \qquad y = 4e^{\frac{t}{2}}, \qquad 0 \le t \le 1$$

$$\frac{dx}{dt} = e^t - 1, \quad \frac{dy}{dt} = 2e^{\frac{t}{2}}$$

Since we are rotating about the **y – axis** the surface area formula is: (because the **x** function is now the radius)

$$\boldsymbol{S = \int_\alpha^\beta 2\pi \cdot x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

$$S = \int_0^1 2\pi(e^t - t)\sqrt{(e^t-1)^2 + \left(2e^{\frac{t}{2}}\right)^2}\,dt \quad (simplify\ under\ the\ radical)$$

$$= 2\pi\int_0^1 (e^t - t)\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,dt = 2\pi\int_0^1 (e^t - t)\sqrt{(e^{2t} + 2e^t + 1)}\,dt$$

$$= 2\pi\int_0^1 (e^t - t)\sqrt{(e^t+1)^2}\,dt = 2\pi\int_0^1 (e^t - t)(e^t + 1)\,dt = 2\pi\int_0^1 (e^{2t} + e^t - te^t - t)dt$$

Now integrate each term. First term use **u – substitution**, the third term use integration by parts.

$$= 2\pi\left[\frac{1}{2}e^{2t} + e^t - (t-1)e^t - \frac{1}{2}t^2\right]_0^1$$

$$\boldsymbol{S = \pi(e^2 + 2e - 6)}$$