3.8 Exponential Growth and Decay

In this section we will study different models for growth and decay.

If we assume that quantities grow or decay at a rate proportional to their size, then if we start at

$y = f(t)$ is the number of individuals in a population at time t, then

$f'(t) = k \cdot f(t)$ (where k is a constant) is the rate of growth proportional to the population $f(t)$.

Many real life situations can be modeled with the equation $f'(t) = k \cdot f(t)$.

Notice that we can rewrite this equation as $\boxed{\dfrac{dy}{dt} = k \cdot y}$ where k is a constant.

If $k > 0$, then it is called the ***law of natural growth.***
If $k < 0$, then it is called the ***law of natural decay.***

This is our ***first*** differential equation because it involves an unknown function y and its derivative $\frac{dy}{dt}$.

Notice that any exponential function of the form $\boxed{y(t) = Ce^{kt}}$, where C is a constant, satisfies $\frac{dy}{dt} = k \cdot y$.

$$y'(t) = k \cdot \left(Ce^{kt}\right) = k \cdot y(t)$$

What happens at $t = 0$? $\quad y(0) = Ce^{k \cdot 0} = C \quad$ Therefore C is the initial value of the function.

Theorem: The only solutions of the differential equation $\frac{dy}{dt} = k \cdot y$ are the exponential functions

$$y(t) = y(0) \cdot e^{kt}$$

One application is seen in population growth.

If we represent the size of a population at time t by $P(t)$, then we can write $\frac{dP}{dt} = k \cdot P$ (solving for k) we get $\frac{1}{P} \cdot \frac{dP}{dt} = k$ which shows that the growth (or decay) rate is divided by the population size. We call this the ***relative growth rate***.

Now notice that using $y(t) = y(0) \cdot e^{kt}$, if the population at time zero ($t = 0$) is represented by P_0, then the population at time t is $\boxed{P(t) = P_0 \cdot e^{kt}}$.

Example: The population of soccer fans in the United States of America was 12.48 million in 1990 and 47.79 million in 2014. (Assume that the growth rate is proportional to the population size.)

a) What is the relative growth rate?

b.) Estimate the population of soccer fans in the United States in 2018.

a) Let $t = 0$ be the time in the year 1990

then, $\mathrm{P}(0) = 12.48$ million and $\mathrm{P}(24) = 47.79$ million

Using $P(t) = P_0 e^{kt}$ we can find $\boldsymbol{k}$.

$P(0) = 12.48e^{k(o)}$

$P(0) = 12.48$

$P(24) = 12.48e^{k\cdot 24} = 47.79$ (solve for $\boldsymbol{k}$)

$e^{24k} = \frac{47.79}{12.48}$ (take ln of both sides)

$\ln e^{24k} = \ln \frac{47.79}{12.48}$

$24k = \ln \frac{47.79}{12.48}$

$\boldsymbol{k = \frac{1}{24} \ln \frac{47.79}{12.48} \approx 0.0559}$ The relative growth rate is about **5.6%** per year.

b) The model is $\boldsymbol{P(t) = 12.48e^{.0559t}}$, when $\boldsymbol{t = 28}$ or year 2018, the population of soccer fans is

$P(28) = 12.48e^{.0559(28)}$

$\boldsymbol{P(28) \approx 59.78}$ **million people**

I hope you remember that these applications were also covered in **pre-calculus**. Please practice the radioactive decay and interest problems.