Limits and Continuity

In Calculus I, we learned the concept of **continuity**. If we have a function $y = f(x)$, and if we have a specified value of x, such as a, we can ask whether or not the function is **continuous** at a. In order for the function to be continuous at a, *all three* of the following conditions must be met:

1. $f(a)$ must be defined. In other words, a must be in the domain of f.

2. $\lim_{x\to a} f(x)$ must exist.

3. $\lim_{x\to a} f(x)$ must be equal to $f(a)$, i.e., $\lim_{x\to a} f(x) = f(a)$.

If *any* of these three conditions is not met, then the function is *not continous* (or is **discontinuous**) at a. In this case, we may say the function has a **discontinuity** at a.

Bear in mind, we adhere to the following conventions:

- The domain of the function excludes any values of x for which $f(x)$ is imaginary or undefined. For instance, the domain of $f(x) = \sqrt{x}$ excludes all negative values of x, and the domain of $f(x) = \frac{1}{x-3}$ excludes 3.
- When we say that a limit "exists," we mean it has a *real-number* value, *not* an imaginary value and *not* an infinite value. Thus, $\lim_{x\to 2} \sqrt{1-x}$ does not exist, and $\lim_{x\to 0} \frac{1}{x^2}$ does not exist. We can say $\lim_{x\to 0} \frac{1}{x^2} = \infty$, but this does not alter the fact that $\lim_{x\to 0} \frac{1}{x^2}$ does not exist!

Books or teacher may sometimes cite only the third condition listed above. Their thinking is that saying $\lim_{x\to a} f(x) = f(a)$ presupposes both condition #1 and condition #2. However, I believe it is best to think of it as three separate conditions, and to check them in the order I have specified. First check condition #1; if it fails, go no further. If condition #1 is met, then check condition #2; if it fails, go no further. If condition #2 is met, then check condition #3.

If we know in advance (based on some previously established theorem) that the function f is continuous at a value a, then we can evaluate $\lim_{x\to a} f(x)$ by simple "plug and chug," i.e., by simply evaluating $f(a)$.

For instance, we have a theorem that says a polynomial function is continuous for all real values of x. Hence, to evaluate $\lim_{x\to 5} (3x^2 - 7x + 4)$, we just plug in 5 for x, giving us 44.

We also have a theorem that says a rational function (i.e., a ratio of polynomials) is continuous for any value of x where the denominator is nonzero. Hence, to evaluate $\lim_{x\to 6} \frac{x+15}{x-3}$, we just plug in 6 for x, giving us 7.

If $f(x)$ is not continuous at a, then $\lim_{x\to a} f(x)$ cannot be evaluated by plug and chug, but this does not mean the limit doesn't exist. It may or may not exist. We have to explore further.

For instance, you cannot evaluate $\lim_{x\to 0} \frac{1}{x}\sin x$ by plug and chug, because $f(x) = \frac{1}{x}\sin x$ has a discontinuity at 0. Instead, we may use L'Hospital's Rule: $\lim_{x\to 0} \frac{1}{x}\sin x = \lim_{x\to 0} \cos x = 1$. (Alternatively, we could apply the Squeeze Theorem, since the function $f(x)$ is sandwiched between the functions $y = 1$ and $y = \cos x$.)

Limits at discontinuities can sometimes be evaulated via algebraic manipulations.

For instance, to find $\lim_{x\to 5} \frac{x^2 - 7x + 10}{x^2 - 25}$, we may reduce the ratio to $\frac{x-2}{x+5}$, so $\lim_{x\to 5} \frac{x^2 - 7x + 10}{x^2 - 25} = \lim_{x\to 5} \frac{x-2}{x+5} = \frac{3}{10}$.

Similarly, to find $\lim_{x\to 1} \frac{x^2 - 1}{\sqrt{2x+3} - \sqrt{5}}$, we may rationalize the denominator and then reduce the ratio to $\frac{(x+1)(\sqrt{2x+3} - \sqrt{5})}{2}$, so $\lim_{x\to 1} \frac{x^2 - 1}{\sqrt{2x+3} - \sqrt{5}} = \lim_{x\to 1} \frac{(x+1)(\sqrt{2x+3} - \sqrt{5})}{2} = 0$.

A **piecewise-defined function** is a function defined not by one formula, but by two or more formulas, where each formula applies to a specified interval and the intervals are non-overlapping. For example, consider the function $f(x) = x^2$ for $x \in (-\infty, 3)$, $-\frac{2}{3}x + 6$ for $x \in [3, \infty)$. This function is piecewise-defined. Its domain is $(-\infty, \infty)$. Bear in mind, it is not two functions, it is *one* function, but the function is defined by two different formulas. The formula x^2 applies applies when $x < 3$. The formula $-\frac{2}{3}x + 6$ applies when $x \geq 3$. Because the intervals $(-\infty, 3)$ and $[3, \infty)$ are non-overlapping, our definition will never produce two values of y from one value of x (in other words, it is a legitimate function). Its graph will pass the vertical line test. Its graph consists of part of the parabola $y = x^2$ "spliced together" with part of the line $y = -\frac{2}{3}x + 6$. Note that $f(3) = 4$. When drawing the graph of the function, we will plot a "solid point" at $(3, 4)$ and an "open point" at $(3, 9)$. The curve comes infinitely close to the point $(3, 9)$, but the point $(3, 9)$ itself is not actually a point on the graph (which is why we depict it as an open point). The function is said to have a **jump discontinuity** (or, for short, a **jump**) at $x = 3$.

In the above example, we may refer to the x value 3 as a "transition point." For a piecewise-defined function, a **transition point** is a point in the domain where the function's definition *transitions* from one formula to another. The function may or may not have a discontinuity at a transition point. In the above example, $f(x)$ had a discontinuity at its transition point. However, in the case of $g(x) = x^3$ for $x \in (-\infty, 2]$, $7x - 6$ for $x \in (2, \infty)$, the function has transition point 2, and it is continuous at that point.

An ordinary limit may be referred to as a **two-sided limit**. We can also consider **one-sided limits**. For any point a in the domain of the function, we may consider the limit to the function as x approaches a from the *left*, denoted $\lim_{x\to a-}$, and we may consider the limit to the function as x approaches a from the *right*, denoted $\lim_{x\to a+}$.

For the above function $f(x)$, $\lim_{x\to 3-} f(x) = 9$, and $\lim_{x\to 3+} f(x) = 4$.

For the above function $g(x)$, $\lim_{x\to 2-} g(x) = 8$, and $\lim_{x\to 2+} g(x) = 8$.

The limit (i.e., the two-sided limit) exists if and only if both of the corresponding one-sided limits exist and their values are equal; when this is the case, the value of the limit is the common value of both one-sided limits.

For the above function $f(x)$, $\lim_{x\to 3} f(x)$ does not exist, because $\lim_{x\to 3-} f(x) \neq \lim_{x\to 3+} f(x)$.

For the above function $g(x)$, $\lim_{x\to 2} g(x)$ exists, because $\lim_{x\to 2-} g(x) = \lim_{x\to 2+} g(x)$, and so $\lim_{x\to 2} g(x) = \lim_{x\to 2-} g(x) = \lim_{x\to 2+} g(x) = 8$.

We now adapt these concepts to deal with a real-valued function with a two-dimensional domain. (In other words, the domain is either the entire x,y plane or some subset of the x,y plane.)

If we have a function $z = f(x,y)$, and if we have a specified point (a,b) in the x,y plane, we can ask whether or not the function is **continuous** at (a,b). In order for the function to be continuous at (a,b), *all three* of the following conditions must be met:

1. $f(a,b)$ must be defined. In other words, (a,b) must be in the domain of f.
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ must exist.
3. $\lim_{(x,y)\to(a,b)} f(x,y)$ must be equal to $f(a,b)$, i.e., $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

If *any* of these three conditions is not met, then the function is *not continous* (or is **discontinuous**) at (a,b). In this case, we may say the function has a **discontinuity** at (a,b).

A polynomial function in x and y is continuous on the entire x,y plane. Hence, limits of polynomial functions may be evaluated by "plug and chug." For instance, $\lim_{(x,y)\to(2,3)} (x^2 - 5xy + y^2) = -17$.

A rational function in x and y is continuous at all points (x,y) where the denominator is nonzero. Hence, limits at such points may be evaluated by "plug and chug." For instance, $\lim_{(x,y)\to(7,2)} \frac{x^2 - 5y}{2x + y^3} = \frac{39}{22}$.

A square root function in x and y is continuous in any region where the radicand is nonnegative. For example $f(x,y) = \sqrt{xy}$ is continuous in the first and third quadrants. Thus, $\lim_{(x,y)\to(6,10)} \sqrt{xy}$ may be found by "plug and chug," giving us $\sqrt{60}$ or $2\sqrt{15}$.

More generally, a *composition* of continuous functions is continuous. Let us make this idea more precise. Suppose h is a function of one variable, and is continuous on given set S. Suppose $g(x,y)$ is a continuous for all (x,y) in a given region R of the x,y plane. Suppose that for every $(x,y) \in R$, $g(x,y) \in S$. Then the composite function $f(x,y) = h(g(x,y))$ is continuous for all (x,y) in the region R.

Another way to state the above idea is this: If the function $g(x,y)$ is continuous at the point (a,b), and if the function h is continuous at the point $g(a,b)$, then the function $h(g(x,y))$ is continuous at the point (a,b).

For instance, $h(u) = \ln u$ is continuous for all $u \in (0, \infty)$. The function $g(x,y) = x^2 + y^2 + 4$ (which is a circular paraboloid) continuous on the entire x,y plane. For every (x,y) in the x,y plane, $g(x,y) \in (0, \infty)$. Then the function $h(g(x,y)) = \ln(x^2 + y^2 + 4)$ is continuous on the x,y plane.

Thus, $\lim_{(x,y)\to(-6,0)} \ln(x^2 + y^2 + 4)$ may be found by "plug and chug," giving us $\ln 40$, which is about 3.6889.

If $f(x,y)$ is not continuous at (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y)$ cannot be evaluated by plug and chug, but this does not mean the limit doesn't exist.

To find $\lim_{(x,y)\to(2,1)} \frac{x^2 + 3xy - 10y^2}{2x^2 - xy - 6y^2}$, we may reduce the ratio to $\frac{x + 5y}{2x + 3y}$, so $\lim_{(x,y)\to(2,1)} \frac{x^2 + 3xy - 10y^2}{2x^2 - xy - 6y^2} = \lim_{(x,y)\to(2,1)} \frac{x + 5y}{2x + 3y} = 1.$

To find $\lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}}$, we may rationalize the denominator and then reduce the ratio to $y(\sqrt{x} + 2\sqrt{y})$, so $\lim_{(x,y)\to(4,1)} \frac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}} = \lim_{(x,y)\to(4,1)} y(\sqrt{x} + 2\sqrt{y}) = 4.$

If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then when we approach the point (a,b) from any direction, we will obtain the same limiting value. Furthermore, we are not restricted to approaching the point (a,b) along a straight path. We should be able to approach (a,b) along *any* path, whether linear or nonlinear, and still approach the same limiting value.

The limit to $f(x,y)$ as we approach the point (a,b) along a specified curve in the x,y plane is known as a **path-specific limit**. (The point (a,b) must lie on the specified curve.) The curve may be a line (which we think of as a *straight* curve). These kinds of limits are often easy to evaluate, by means of an algebraic substitution.

For instance, the limit to $f(x,y) = \frac{x^2 + 2xy + y^2}{x^2 + y^2}$ as (x,y) approaches $(0,0)$ along the line $y = 3x$ may be found by substituting $3x$ in place of y. Then $f(x,y) = \frac{x^2 + 2x(3x) + (3x)^2}{x^2 + (3x)^2} = \frac{16x^2}{10x^2} = \frac{8}{5}$, so the limit is $\frac{8}{5}$. On the other hand, the limit to this function as (x,y) approaches $(0,0)$ along the line $y = -2x$ may be found by substituting $-2x$ in place of y. Then $f(x,y) = \frac{x^2 + 2x(-2x) + (-2x)^2}{x^2 + (-2x)^2} = \frac{x^2}{5x^2} = \frac{1}{5}$, so the limit is $\frac{1}{5}$. More generally, the limit to this function as (x,y) approaches $(0,0)$ along the line $y = mx$ may be found by substituting mx in place of y. Then $f(x,y) = \frac{x^2 + 2x(mx) + (mx)^2}{x^2 + (mx)^2} = \frac{x^2 + 2mx^2 + m^2x^2}{x^2 + m^2x^2} = \frac{1 + 2m + m^2}{1 + m^2}$, so the limit is $\frac{1 + 2m + m^2}{1 + m^2}$. For different values of m, we get different limits.

In order for $\lim_{(x,y)\to(a,b)} f(x,y)$ to exist, all possible path-specific limits as (x,y) approaches (a,b) must exist, and they must all have the same value; when this is the case, the value of the limit is the common value of all path-specific limits. Note: When we refer to "all possible path-specific limits as (x,y) approaches (a,b)," we mean the limits as (x,y)

approaches (a, b) along all possible paths–i.e., along every possible curve containing the point (a, b).

$\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist if any path-specific limit does not exist, or if any two path-specific limits have different values.

Thus, $\lim_{(x,y)\to(a,b)} \frac{x^2 + 2xy + y^2}{x^2 + y^2}$ does not exist.

Recall that in Calculus I, $\lim_{x\to a} f(x)$ may be thought of as a *two-sided limit*, and it exists if and only if both one-sided limits exist and are equal. Analagously, $\lim_{(x,y)\to(a,b)} f(x,y)$ may be thought of as an **all-path limit**, and it exists if and only all path-specific limits exist and are equal.

Consider $f(x,y) = \frac{3xy^2}{x^2 + y^4}$.

- To find the limit to this function as (x,y) approaches $(0,0)$ along any line $y = mx$, we substitute mx in place of y, giving us $\frac{3xm^2}{1 + m^4x^2}$, which approaches 0 as x approaches 0, regardless of the value of m.
- To find the limit to this function as (x,y) approaches $(0,0)$ along any parabola $y = ax^2$, we substitute ax^2 in place of y, giving us $\frac{3a^2x^5}{1 + a^4x^6}$, which approaches 0 as x approaches 0, regardless of the value of a.
- To find the limit to this function as (x,y) approaches $(0,0)$ along any parabola $x = by^2$, we substitute by^2 in place of x, giving us $\frac{3b}{b^4 + 1}$, and so this path-specific limit has a value of $\frac{3b}{b^4 + 1}$, which obviously depends on b. In other words, for different values of b, we get different limits. For instance, when $b = 1$, i.e., when we approach along the parabola $x = y^2$, we get a limit of $\frac{3}{2}$, but when $b = -1$, we get a limit of $-\frac{3}{2}$.

Consequently, $\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2 + y^4}$ does not exist.

All the above concepts can be adapted to deal with a real-valued function with a three-dimensional domain.

If we have a function $w = f(x,y,z)$, and if we have a specified point (a,b,c) in x,y,z space, we can ask whether or not the function is **continuous** at (a,b,c). In order for the function to be continuous at (a,b,c), *all three* of the following conditions must be met:

1. $f(a,b,c)$ must be defined. In other words, (a,b,c) must be in the domain of f.
2. $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)$ must exist.
3. $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z)$ must be equal to $f(a,b,c)$, i.e., $\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$.

If *any* of these three conditions is not met, then the function is *not continous* (or is **discontinuous**) at (a,b,c). In this case, we may say the function has a **discontinuity** at (a,b,c).