Two-Dimensional Curves

A **curve** in two-dimensional space (i.e., in the x,y coordinate system) may be represented algebraically by an equation involving the variables x and y. If we can solve the equation for y in terms of x, in such a way that any *one* value of x generates *one* value of y, then the curve is the graph of a **function**–i.e., y is a function of x, $y = f(x)$. Graphically, this means the curve passes the **Vertical Line Test**: Any vertical line will intersect the graph at *no more than* one point.

For instance, given the equation $2x - 3y = 6$, if we solve for y, we get $y = \frac{2}{3}x - 2$, so y is a function of x, i.e., $y = f(x) = \frac{2}{3}x - 2$. The graph is a line with slope $\frac{2}{3}$ and y intercept $(0, -2)$. (A line may be thought of as a special case of a curve, namely, a curve that is *straight*.) This is an example of a **linear function**. (Any *oblique*–i.e., slanted–line represents a linear function. A *horizontal* line represents a **constant function**. A *vertical* line does not represent a function at all.)

Given the equation $4x^2 + 2y = 10$, if we solve for y, we get $y = -2x^2 + 5$, so y is a function of x, i.e., $y = f(x) = -2x^2 + 5$. The graph is a downward opening **parabola** with y intercept $(0, 5)$. This is an example of a **quadratic function.**

Given an equation in x and y, it may be impossible to solve for y in terms of x, for example, $(x + y)^{\cos(x^5 - y^7)} = 1$. Or it may be possible to solve for y in terms of x, but the resulting formula may generate *more than one* value of y from *one* value of x. For instance, given the equation $x^2 + y^2 = 9$, solving for y gives us $y = \pm\sqrt{9 - x^2}$, so *one* value of x generates *two* values of y (for instance, when $x = 0$ we get $y = 3$ and $y = -3$). In such cases, y is *not* a function of x. Graphically, this means the curve fails the Vertical Line Test: A vertical line may intersect the graph at *more than* one point. In the case of $x^2 + y^2 = 9$, the graph is a circle centered at the origin with radius 3. The vertical line $x = 0$ intersects this circle at the points $(0, 3)$ and $(0, -3)$, which are the y intercepts of the graph.

Whenever a curve is represented by an equation involving only x and y, the curve has no *orientation*–i.e., there is no *forward direction* or *backward direction* for the curve. (Orientation will be discussed in detail shortly.)

In Calculus II, we learned that a curve in two-dimensional space may be represented algebraically by a pair of **parametric equations**, which express x and y each in terms of a third variable, known as a **parameter**. We often use t (representing time) or θ (representing angle measure). Assuming the parameter is t, we generically refer to these equations as $x = x(t)$, $y = y(t)$. The process of writing parametric equations to represent a curve is known as **parameterizing** the curve. There is typically more than one way to do this.

When a curve is parameterized with respect to time, t, we think of it as the **path** of a *moving particle*. The parametric equations give a unique **position** for the particle at each point in time. This is known as a **motion paradigm**.

Assuming 0 is in the domain of the parametric equations, the point corresponding to $t = 0$ or $\theta = 0$ is of special interest; it is referred to as the **initial point** or **starting point**, denoted P_0. Different parametrizations of a curve may yield different starting points. For instance, if $x^2 + y^2 = 9$ is parameterized as $x = 3\cos\theta,\ y = 3\sin\theta,$ the starting point is $(3,0),$ but if we use the parameterization $x = -3\cos\theta,\ \ y = 3\sin\theta,$ the starting point is $(-3,0).$

Parameterizing the curve also introduces an **orientation** for the curve–i.e., a forward direction and a backward direction. The **forward direction** is the direction that we move along the curve as the value of the parameter *increases*, whereas the **backward direction** is the direction that we move along the curve as the value of the parameter *decreases*. Different parametrizations of a curve may yield different orientations. For instance, if $x^2 + y^2 = 9$ is parameterized as $x = 3\cos\theta,\ y = 3\sin\theta,$ the orientation is *counter-clockwise* (i.e., the forward direction is counter-clockwise), but if we use the parameterization $x = -3\cos\theta,\ \ y = 3\sin\theta,$ the orientation is *clockwise*.

If $x^2 + y^2 = 9$ is parameterized as $x = 3\cos\theta,\ y = -3\sin\theta,$ the orientation is clockwise and the starting point is $(3,0).$

A circle is an example of a simple closed curve. For such curves, it makes sense to describe orientation or direction as either "clockwise" or "counter-clockwise." For other curves, such as parabolas, these terms are inapplicable. How can we describe direction for such curves? In some cases, we may be able to describe direction as "leftward" or "rightward," but this wouldn't work in all cases (for example, it would work for an upright parabola, but not for a sideways parabola). Another option would be to describe direction as "upward" or "downward," but this again won't work in all cases (for example, it would work for a sideways parabola, but not for an upright parabola).

If we use the parameter t and if 1 is in the domain of the parametric equations, the point corresponding to $t = 1$ is of special interest; it is referred to as the **unitary point** and is denoted P_1. (As with P_0, the point depends on the chosen parameterization.) We can say the forward direction of the curve is the direction *from* P_0 *to* P_1.

The parabola $y = -2x^2 + 5$ could be parameterized so that $x = t,\ \ y = -2t^2 + 5,$ in which case $P_0 = (0,5)$ and $P_1 = (1,3).$ Or it could be parameterized so that $x = t + 4,$ $y = -2(t+4)^2 + 5,$ in which case $P_0 = (4,-27)$ and $P_1 = (5,-45).$ Or it could be parameterized so that $x = -t,\ \ y = -2t^2 + 5,$ in which case $P_0 = (0,5)$ and $P_1 = (-1,3).$ The first two parameterizations give us a rightward orientation, whereas the third gives us a leftward orientation.

We have already discussed how to parameterize a line: Choose two different points, (x_0,y_0) and (x_1,y_1). Let $a = x_1 - x_0,$ and let $b = y_1 - y_0.$ Then the line has parametric equations $x = x_0 + at,\ y = y_0 + bt,$ where $t \in (-\infty,\infty).$ The initial point is $P_0 = (x_0,y_0),$ and the unitary point is $P_1 = (x_1,y_1).$ The forward direction is the direction we follow when moving from P_0 to P_1. Values of t between 0 and 1 give us points on the line between P_0 and P_1.

- If P_1 lies to the right of $P_0,\ t < 0$ gives us points on the line to the left of $P_0,$ and $t > 1$ gives us points on the line to the right of P_1.

- If P_1 lies to the left of P_0, $t < 0$ gives us points on the line to the right of P_0, and $t > 1$ gives us points on the line to the left of P_1.
- If P_1 lies above P_0, $t < 0$ gives us points on the line below P_0, and $t > 1$ gives us points on the line above P_1.
- If P_1 lies below P_0, $t < 0$ gives us points on the line above P_0, and $t > 1$ gives us points on the line below P_1.

For instance, to parameterize the line $2x - 3y = 6$, we may choose $P_0 = (0,2)$ and $P_1 = (3,0)$, so $a = 3$ and $b = -2$. The parametric equations are then $x = 0 + 3t$, $y = 2 - 2t$. The orientation of this line is rightward and downward (or "southeast"). Since P_1 is to the right of and below P_0, $t < 0$ gives us points on the line to the left of and above P_0, while $t > 1$ gives us points on the line to the right of and below P_1.

Suppose a curve has parametric equations $x = x(t)$, $y = y(t)$. For any value of t, we obtain a point on the curve, $P_t = (x(t), y(t))$. The position vector for this point is $\mathbf{r}(t) = < x(t), y(t) >$ $= x(t)\mathbf{i} + y(t)\mathbf{j}$. Notice that this is a **vector-valued function** of one real variable (or parameter), t. It is known as a **position function** for the curve. (Bear in mind, this function depends upon the chosen parameterization for the curve. A given curve can have many possible parameterizations and hence many possible position functions.) Of course, we can use any variable in place of t.

For the line $2x - 3y = 6$, we may use position function $\mathbf{r}(t) = < 3t, 2 - 2t >$. For the parabola $4x^2 + 2y = 10$, we may use position function $\mathbf{r}(t) = < t, -2t^2 + 5 >$. For the circle $x^2 + y^2 = 9$, we may use position function $\mathbf{r}(\theta) = < 3\cos\theta, 3\sin\theta >$.

Limits And Continuity:

In Calculus I, we learned the basic concept of the limit. If we have a function $y = f(x)$, we can ask, does y approach any particular value as x approaches some specified value, such as a. If it does, we say the function has a **limiting value** (or just a **limit**, for short) as x approaches a. Suppose we have such a value. Call it L. We can say, "y approaches L as x approaches a," which can be written more compactly as follows: $y \to L$ as $x \to a$. We can also write $\lim_{x \to a} y = L$, or $\lim_{x \to a} f(x) = L$, which would be pronounced, "The limit to the function $f(x)$ as x approaches a is L." For example, $\lim_{x \to 0} \frac{1}{x} \sin x = 1$.

In Calculus I, we were dealing with functions that produced *numerical values* (i.e., for any given numerical value of x, the function $y = f(x)$ produces a numerical value of y). But now, in Calculus III, we are dealing with functions that produce *vector values*. In other words, the vector equation of a curve, $\mathbf{r}(t) = < x(t), y(t) >$, may be thought of as a function whose *input* is the *scalar* (or real number) t, and whose *output* is the *vector* $\mathbf{r}(t)$. For instance, given the numerical value $t = 2$, the function $\mathbf{r}(t) = < t^3, \frac{1}{t} >$ produces the vector $< 8, \frac{1}{2} >$. Can we apply the concept of the limit to such functions? We can! Here is how...

Given the vector-valued function $\mathbf{r}(t) = < x(t), y(t) >$, we can ask, does $\mathbf{r}(t)$ approach any particular *vector value* as t approaches some specified *numerical value*, such as a. If it does, we say the function has a **limiting** (**vector**) **value** (or just a **limit**, for short) as t

approaches a. Suppose we have such a vector value. Call it $\mathbf{L}$. We can say, "$\mathbf{r}(t)$ approaches $\mathbf{L}$ as t approaches a," which can be written more compactly as follows: $\mathbf{r}(t) \to \mathbf{L}$ as $t \to a$. We can also write $\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}$, which would be pronounced, "The limit to the function $\mathbf{r}(t)$ as t approaches a is $\mathbf{L}$."

That's the basic idea. Now how do we go about finding this sort of limit? We use the following principle...

- If $\mathbf{r}(t) = < x(t), y(t) >$, then $\lim_{t\to a} \mathbf{r}(t) = < \lim_{t\to a} x(t), \lim_{t\to a} y(t) >$
- Equivalently, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} x(t)\mathbf{i} + \lim_{t\to a} y(t)\mathbf{j}$

Essentially, this says the limit distributes in the same way as scalar multiplication–recall that $c\mathbf{r}(t) = < cx(t), cy(t) >$.

Each of the limits on the right side of the equation can be evaluated using the methods of Calculus I.

Suppose $\mathbf{r}(t) = \frac{1}{t}\sin t\ \mathbf{i} + \frac{t^2+6t}{t^2-2t}\ \mathbf{j}$. Find $\lim_{t\to 0} \mathbf{r}(t)$.
Solution: $\lim_{t\to 0} \mathbf{r}(t) = \lim_{t\to 0} \frac{1}{t}\sin t\ \mathbf{i} + \lim_{t\to 0} \frac{t^2+6t}{t^2-2t}\ \mathbf{j} = \mathbf{i} - 3\mathbf{j}$

As we learned in Calculus I, some limits do not exist. For instance, $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist, due to infinite oscillation. A limit "not existing" includes the possibility that the function could approach infinity. For instance, $\lim_{x\to 0} \frac{1}{x^2}$ does not exist, because the function approaches infinity as x approaches 0. We may write $\lim_{x\to 0} \frac{1}{x^2} = \infty$, but it is still the case that the limit does not exist! (Saying the limit "exists" means the function approaches a unique real number value, and ∞ is not a real number.)

Likewise, in this new situation, a limit may or may not exist. In order for $\lim_{t\to a} \mathbf{r}(t)$ to exist, *both* limits on the right side of the equation must exist. In other words, $\lim_{t\to a} \mathbf{r}(t)$ exists if and only if *both* of the following limits exist:

- $\lim_{t\to a} x(t)$
- $\lim_{t\to a} y(t)$

If *either* of these does not exist, then $\lim_{t\to a} \mathbf{r}(t)$ does not exist.

For instance, suppose $\mathbf{r}(t) = < 5t+2, \frac{1}{t-3} >$. $\lim_{t\to 3} \mathbf{r}(t)$ does not exist, because $\lim_{t\to 3} \frac{1}{t-3}$ does not exist.

In Calculus I, we learned the concept of **continuity**. If we have a function $y = f(x)$, and if we have a specified value of x, such as a, we can ask whether or not the function is **continuous** at a. In order for the function to be continuous at a, *all three* of the following conditions must be met:

1. $f(a)$ must be defined. In other words, a must be in the domain of f.
2. $\lim_{x\to a} f(x)$ must exist.
3. $\lim_{x\to a} f(x)$ must be equal to $f(a)$, i.e., $\lim_{x\to a} f(x) = f(a)$.

If *any* of these three conditions is not met, then the function is *not continous* (or is **discontinuous**) at a. In this case, we may say the function has a **discontinuity** at a.

Books or teachers may sometimes cite only the third condition listed above. Their thinking is that saying $\lim_{x\to a} f(x) = f(a)$ presupposes both condition #1 and condition #2. However, I believe it is best to think of it as three separate conditions, and to check them in the order I have specified. First check condition #1; if it fails, go no further. If condition #1 is met, then check condition #2; if it fails, go no further. If condition #2 is met, then check condition #3.

If we know in advance (based on some previously established theorem) that the function f is continuous at a value a, then we can evaluate $\lim_{x\to a} f(x)$ by simple "plug and chug," i.e., by simply evaluating $f(a)$. For instance, we have a theorem that says a polynomial function is continuous for all real values of x. Hence, to evaluate $\lim_{x\to 5} (3x^2 - 7x + 4)$, we just plug in 5 for x, giving us 44. But be careful. Plug and chug does not work when the function is not continuous! For instance, you cannot evaluate $\lim_{t\to 0} \frac{1}{t} \sin t$ by plug and chug.

The concept of continuity can be applied to vector-valued functions. The function $\mathbf{r}(t) = < x(t), y(t) >$ is **continuous** at $t = a$ if and only if *all three* of the following conditions are met:

1. $\mathbf{r}(a)$ must be defined, which means $x(a)$ and $y(a)$ must both be defined. In other words, a must be in the domain of each function.
2. $\lim_{t\to a} \mathbf{r}(t)$ must exist, which means $\lim_{t\to a} x(t)$ and $\lim_{t\to a} y(t)$ must both exist.
3. $\lim_{t\to a} \mathbf{r}(t)$ must equal $\mathbf{r}(a)$, which means $\lim_{t\to a} x(t) = x(a)$ and $\lim_{t\to a} y(t) = y(a)$.

If *any* of these three conditions is not met, then the function is *not continous* (or is **discontinuous**) at a. In this case, we may say the function has a **discontinuity** at a.

Derivatives:

If a curve has a nonvertical tangent line at a certain point, the slope of the tangent line is obtained by **differentiation**. When y is a function of x, we use *ordinary* differentiation, which gives us the **derivative** of y with respect to x, $f'(x)$ or $\frac{dy}{dx}$, in terms of x. Specifically, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. The quantity $\frac{f(x+h) - f(x)}{h}$ is known as the **difference quotient**. It represents the slope of the **secant line** passing through a fixed point $(x, f(x))$ and a variable point $(x + h, f(x + h))$. As h approaches zero, the latter point approaches the former point and the secant line approaches the tangent line. (Of course, in practice, we find the derivative by using the *rules of differentiation* studied in Calculus I.)

In the case of a constant function or a linear function, the graph of the function is already a line, so the tangent line coincides with this line itself; hence, the derivative at any point is simply the slope of the original line (which means the derivative is a fixed value–i.e., it does not vary as x varies). For other functions, the derivative is *not* fixed, but rather varies as x varies. For instance, in the case of $y = f(x) = -2x^2 + 5$, the derivative is $-4x$ (so the slope of the tangent line is 12 when x is -3, whereas the slope of the tangent line is -20 when x is 5). In such cases, the derivative is a function of *one* variable, x.

When y is *not* a function of x, we may find the derivative by using **implicit differentiation**: We differentiate both sides of the equation with respect to x, and then solve the resulting equation for $\frac{dy}{dx}$ in terms of both x and y. For instance, in the case of $x^2+y^2=9$, we get $\frac{dy}{dx}=-\frac{x}{y}$. In such cases, the derivative is a function of *two* variables, x and y. In the case of $x^2+y^2=9$, consider two points that vertically align with each other, $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2},\frac{-\sqrt{2}}{2})$. At the former, we get $\frac{dy}{dx}=-1$, and at the latter, we get $\frac{dy}{dx}=1$. Thus, the equation of the tangent line at the former point is $y=-x+\sqrt{2}$, whereas the equation of the tangent line at the latter point is $y=x-\sqrt{2}$.

When the curve has been parameterized, if the curve has a nonvertical tangent line at a certain point, the slope of the tangent line may be obtained by **parametric differentiation**, which gives us the slope in terms of the parameter. Suppose our parameter is t. Generically, our parametric equations are $x=x(t),\ \ y=y(t)$. Rather than expressing $\frac{dy}{dx}$ as a function of x (as we would do in the case of ordinary differentiation), or as a function of both x and y (as we would do in the case of implicit differentiation), we instead express it as a function of t. First, we find the derivatives $\frac{dx}{dt}=x'(t)$ and $\frac{dy}{dt}=y'(t)$. We then divide the latter by the former, and the result is the slope of the tangent line, i.e., $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{y'(t)}{x'(t)}$. Obviously, all these equations are valid if we replace t with a different parameter, such as θ.

Suppose the circle $x^2+y^2=9$ is parameterized as $x=3\cos\theta,\ y=3\sin\theta$. Then $\frac{dx}{d\theta}=-3\sin\theta$ and $\frac{dy}{d\theta}=3\cos\theta$. When $\theta=\frac{\pi}{6}$, we get the point $(\frac{3\sqrt{3}}{2},\frac{3}{2})$, and we get $\frac{dx}{d\theta}=-\frac{3}{2}$ and $\frac{dy}{d\theta}=\frac{3\sqrt{3}}{2}$, so $\frac{dy}{dx}=-\sqrt{3}$. (This is the same result we would have obtained via implicit differentiation.)

A vector-valued function of one parameter can be **differentiated** as follows. Assuming the parameter is t, we define the **derivative** with respect to t to be $\lim_{h\to 0}\frac{1}{h}(\mathbf{r}(t+h)-\mathbf{r}(t))$, which equals $\lim_{h\to 0}\frac{x(t+h)-x(t)}{h}\mathbf{i}+\lim_{h\to 0}\frac{y(t+h)-y(t)}{h}\mathbf{j}$, or $x'(t)\mathbf{i}+y'(t)\mathbf{j}$, or $\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}$. In component form, this is $< x'(t), y'(t) >$ or $< \frac{dx}{dt},\frac{dy}{dt} >$. The derivative is denoted $\mathbf{r}'(t)$ or $\frac{d}{dt}\mathbf{r}(t)$ or $\frac{d\mathbf{r}}{dt}$.

When $\mathbf{r}(t)$ represents a position function and the parameter t represents *time*, then $\mathbf{r}'(t)$ is interpreted as the **velocity function**, in which case we may write $\mathbf{v}(t)$ in place of $\mathbf{r}'(t)$.

Notice that $\mathbf{r}'(t)$ or $\mathbf{v}(t)$ is a vector. For the moment, assume this vector is nonzero. Then it has a magnitude or length, which is a positive real number, and it has a direction. The *magnitude* of velocity is the **speed** of motion, $v(t)=\sqrt{x'(t)^2+y'(t)^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$. The *direction* of velocity represents the *instantaneous direction of motion.* This direction will be along the curve's tangent line at the given point. (To write the equation of the tangent line, we use the velocity vector as the line's direction vector.)

For example, for the parabola $x=t,\ \ y=-2t^2+5$, we have position function $\mathbf{r}(t)=\ < t,-2t^2+5 >$, velocity function $\mathbf{v}(t)=\ < 1,-4t >$, and speed function $v(t)=\sqrt{1+16t^2}$. When $t=3$, we have the point $(3,-13)$, the position vector $< 3,-13 >$, the velocity vector $< 1,-12 >$, and speed $\sqrt{145}\approx 12.04$. At the point $(3,-13)$, the tangent line has parametric equations $x=3+t,\ y=-13-12t$.

Suppose that at a certain value of t, $x'(t) = 0$ and $y'(t) \neq 0$, so $\mathbf{v}(t)$ is a nonzero scalar multiple of $\mathbf{j}$. In this case, the curve has a **vertical tangent line** at the point in question. On the other hand, suppose $y'(t) = 0$ and $x'(t) \neq 0$, so $\mathbf{v}(t)$ is a nonzero scalar multiple of $\mathbf{i}$. In this case, the curve has a **horizontal tangent line** at the point in question. If $x'(t)$ and $y'(t)$ are both nonzero, the curve has an **oblique** (**or slanted**) **tangent line** at the point in question. If $x'(t) = 0$ and $y'(t) = 0$ (in other words, if the velocity vector is the zero vector), then the curve has a **cusp** or **kink** or **sharp turn** at the point in question. (In this case, it may or may not have a tangent line at the point in question, depending on whether the *left-hand tangent* coincides with the *right-hand tangent*.)

For the circle $x = 3\cos\theta,\ y = 3\sin\theta,$ we have $\mathbf{r}'(\theta) = < -3\sin\theta, 3\cos\theta >$. $\mathbf{r}'(0) = < 0,3 > = 3\mathbf{j}$ and $\mathbf{r}'(\pi) = < 0,-3 > = -3\mathbf{j},$ so the circle has vertical tangent lines when $\theta = 0$ and $\theta = \pi,$ i.e., at the points $(3,0)$ and $(-3,0)$. $\mathbf{r}'(\frac{\pi}{2}) = < -3,0 > = -3\mathbf{i}$ and $\mathbf{r}'(\frac{3\pi}{2}) = < 3,0 > = 3\mathbf{i},$ so the circle has horizontal tangent lines when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$, i.e., at the points $(0,3)$ and $(0,-3)$. $\mathbf{r}'(\theta)$ is never zero because the sine and cosine functions are never simultaneously zero; hence the circle has no kinks.

The function $y = x^{2/3}$ can be parameterized as $x = t^3,\ y = t^2,$ which gives us $\mathbf{v}(t) = < 3t^2, 2t >$. Since $\mathbf{v}(0) = \mathbf{0}$, the curve has a kink when $t = 0$, i.e., at the point $(0,0)$. In this case, we have a vertical tangent line at the origin. (Note: In Calculus I, if we differentiated the equation $y = x^{2/3}$, we would get $\frac{dy}{dx} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, which is undefined when x is zero.)

Since $\mathbf{r}'(t)$ is itself a vector-valued function of t, it may likewise be differentiated with respect to t. The result is the **second derivative** of $\mathbf{r}(t)$ with respect to t, $\mathbf{r}''(t)$ or $\frac{d^2\mathbf{r}}{dt^2}$. It is equal to $x''(t)\mathbf{i} + y''(t)\mathbf{j}$, or $\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$, or $< x''(t), y''(t) >$, or $< \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} >$.

When $\mathbf{r}(t)$ represents a position function and t represents time, $\mathbf{r}''(t)$ is interpreted as the **acceleration function**, in which case we may write $\mathbf{a}(t)$ in place of $\mathbf{r}''(t)$. Its magnitude is $a(t) = \sqrt{x''(t)^2 + y''(t)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$.

See the **Differentiation Rules** on page 858 of the text.

One rule the text neglects to mention is this: For any constant vector $\mathbf{C}$, $\frac{d}{dt}\mathbf{C} = \mathbf{0}$

Integrals:

Given two vector-valued functions $\mathbf{u}(t)$ and $\mathbf{w}(t)$, if $\mathbf{u}'(t) = \mathbf{w}(t)$ for all t in an open interval, then $\mathbf{w}(t)$ is the **derivative** of $\mathbf{u}(t)$, and $\mathbf{u}(t)$ is an **antiderivative** of $\mathbf{w}(t)$. Pay close attention to the wording just used. We say "the" derivative because $\mathbf{u}(t)$ has a *unique* derivative, but we say "an" antiderivative because $\mathbf{w}(t)$ will have *infinitely many* antiderivatives. For any constant vector $\mathbf{C}$, the function $\mathbf{u}(t) + \mathbf{C}$ is an antiderivative of $\mathbf{w}(t)$, because $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{C}) = \frac{d}{dt}\mathbf{u}(t) + \frac{d}{dt}\mathbf{C} = \mathbf{u}'(t) + \mathbf{0} = \mathbf{u}'(t) = \mathbf{w}(t)$. The collection of *all* antiderivatives of $\mathbf{w}(t)$ is called the **indefinite integral** of $\mathbf{w}(t)$ and is denoted $\int \mathbf{w}(t)\, dt$. We may write $\int \mathbf{w}(t)\, dt = \mathbf{u}(t) + \mathbf{C}$, where $\mathbf{C}$ is an **arbitrary constant vector**. The indefinite integral of $\mathbf{w}(t)$ can also be referred to as the **general antiderivative** of $\mathbf{w}(t)$.

The arbitrary constant vector can be expressed as $< C_1, C_2 >$.

For example, consider $\mathbf{u}(t) = < t^2, \sin t >$ and $\mathbf{w}(t) = < 2t, \cos t >$. Since $\mathbf{u}'(t) = \mathbf{w}(t)$ for all $t \in (-\infty, \infty)$, $\mathbf{w}(t)$ is the derivative of $\mathbf{u}(t)$, and $\mathbf{u}(t)$ is an antiderivative of $\mathbf{w}(t)$.
$\int < 2t, \cos t > dt = < t^2, \sin t > + \mathbf{C} = < t^2, \sin t > + < C_1, C_2 > = < t^2 + C_1, \sin t + C_2 >$.

A generic antiderivative of $\mathbf{w}(t)$ can be denoted $\mathbf{W}(t)$.

A **particular antiderivative** can be dictated by an **initial condition**. For instance, suppose we seek the antiderivative of $\mathbf{w}(t) = < 2t, \cos t >$ whose value when $t = \frac{\pi}{2}$ is $< 5, 7 >$. In other words, find $\mathbf{W}(t)$ so that $\mathbf{W}(\frac{\pi}{2}) = < 5, 7 >$. We already know that the general antiderivative of $\mathbf{w}(t)$ is $< t^2 + C_1, \sin t + C_2 >$. Hence, the challenge is to find the necessary values of the constants C_1 and C_2. $\left(\frac{\pi}{2}\right)^2 + C_1 = 5$, so $C_1 = 5 - \frac{\pi^2}{4} = \frac{20-\pi^2}{4}$, and $\sin\frac{\pi}{2} + C_2 = 7$, so $C_2 = 6$. Hence, we want the particular antiderivative $\mathbf{W}(t) = < t^2 + \frac{20-\pi^2}{4}, \sin t + 6 >$.

If $\mathbf{w}(t) = < x(t), y(t) >$, then $\int \mathbf{w}(t)\, dt = \int < x(t), y(t) > dt = < \int x(t)\, dt, \int y(t)\, dt >$, or $\int x(t)\, dt\, \mathbf{i} + \int y(t)\, dt\, \mathbf{j}$.

For any real numbers a and b, $\int_a^b \mathbf{w}(t)\, dt = \int_a^b < x(t), y(t) > dt = < \int_a^b x(t)\, dt, \int_a^b y(t)\, dt >$, or $\int_a^b x(t)\, dt\, \mathbf{i} + \int_a^b y(t)\, dt\, \mathbf{j}$. This is known as the **definite integral** of $\mathbf{w}(t)$ over the interval on the t axis with endpoints a and b. It gives us a particular vector, rather than a vector-valued function. a and b are known as the **limits or boundaries of integration**.

If $\mathbf{W}(t)$ is any antiderivative of $\mathbf{w}(t)$, then $\int_a^b \mathbf{w}(t)\, dt = \mathbf{W}(b) - \mathbf{W}(a)$. This may be denoted

$[\mathbf{W}(t)]_a^b$