15.6) Triple Integrals

A *rectangular box* is a region $B = \{(x,y,z) \mid x \in [a,b],\ y \in [c,d],\ z \in [p,q]\} = [a,b] \times [c,d] \times [p,q]$, where $a < b,\ c < d$, and $p < q$.

A triple integral over this box is $\iiint_B f(x,y,z)\ dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\ dz\ dy\ dx$. By Fubini's Theorem, this can be rewritten into five other possible orders, such as $\int_c^d \int_p^q \int_a^b f(x,y,z)\ dx\ dz\ dy$, so long as f is continuous on B.

The Integral Factorization Principle: If $f(x,y,z)$ can be factored as $r(x)s(y)t(z)$, then

$$\int_a^b \int_c^d \int_p^q f(x,y,z)\ dz\ dy\ dx = \int_a^b r(x)\ dx \int_c^d s(y)\ dy \int_p^q t(z)\ dz.$$

Example One: $\int_0^1 \int_{-1}^2 \int_0^3 xyz^2\ dz\ dy\ dx = \int_0^1 x\ dx \int_{-1}^2 y\ dy \int_0^3 z^2\ dz = \left[\tfrac{1}{2}x^2\right]_0^1 \left[\tfrac{1}{2}y^2\right]_{-1}^2 \left[\tfrac{1}{3}z^3\right]_0^3 =$
$\tfrac{1}{2}(1-0)\tfrac{1}{2}(4-1)\tfrac{1}{3}(27-0) = (\tfrac{1}{2})(\tfrac{3}{2})(9) = \tfrac{27}{4}$

We now consider triple integrals over finite three-dimensional regions (known as *solids*) other than rectangular boxes.

We begin by reviewing and extending some geometric concepts discussed earlier this semester...

Let C_1 be a given curve in the x,y plane. Let C_2 be its orthogonal projection into x,y,z space. C_2 is a special type of surface known as a **cylinder** (parallel to the z axis). If the curve C_1 is a circle, then the cylinder C_2 is known as a **circular cylinder**. We shall assume C_1 is a simple, closed curve–possibly a circle, possibly an ellipse, possibly a more complicated shape.

The cylinder extends infinitely far in the direction of positive z and in the direction of negative z, but we are interested in a *finite section* of the cylinder, bounded between two horizontal planes, the lower plane $z = p$ and the upper plane $z = q$ (in which case, the finite section has vertical height $q - p$).

The closed three-dimensional region bounded by the cylinder C_2 and the planes $z = p$ and $z = q$ is known as a **cylindrical solid**. This region encompasses both its boundary points and its interior points.

If the curve C_1 is a circle, then the we have a **circular cylindrical solid**. Think of it as a can of baked beans. It has three faces: The *top* face is circular disk, the *bottom* face is a circular disk (congruent to the top face), and the *lateral* or *vertical* face is a circular "tube."

Now suppose that we replace the lower boundary plane $z = p$ with a surface that is the graph of a function, $z = u_1(x,y)$, and we replace the upper boundary plane $z = q$ with another surface that is the graph of a function, $z = u_2(x,y)$, where $u_2(x,y) > u_1(x,y)$. The closed three-dimensional region bounded by the cylinder C_2 and the surfaces $z = u_1(x,y)$ and $z = u_2(x,y)$ may be referred to as a **modified cylindrical solid**. Let us refer to this solid as E.

If the curve C_1 is a circle, then think of the modified cylindrical solid as a can of baked beans where the top and bottom have been warped so they are no longer flat, but the lateral face or tube has not been warped.

We have stipulated that C_1 is a simple, closed curve. Let D be the closed region of the x,y plane bounded by C_1. In other words, D consists of the points on C_1 and the interior points encompassed by C_1.

D is the orthogonal projection of the modified cylindrical solid E onto the x,y plane.

We shall assume D is either a Type I region or a Type II region.

$$\iiint_E f(x,y,z)\ dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz \right] dA.$$

- If D is Type I, we get $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dy\ dx.$
- If D is Type II, we get $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dx\ dy.$

The above triple integrals are known as Type 1 triple integrals. (Note that we use the Arabic numeral 1 instead of the Roman numeral I.) The region E is known as a Type 1 solid.

We can also have Type 2 and Type 3 triple integrals and solids.

- In Type 2, region D is in the y,z plane, and the modified cylindrical solid E is parallel to the x axis, capped off by the surfaces $x = u_1(y,z)$ and $x = u_2(y,z)$.

  $$\iiint_E f(x,y,z)\ dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx \right] dA.$$

- In Type 3, region D is in the x,z plane, and the modified cylindrical solid E is parallel to the y axis, capped off by the surfaces $y = u_1(x,z)$ and $y = u_2(x,z)$.

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA.$$

Here is an example of a Type 1 triple integral. Let C_1 be the circle $x^2 + y^2 = 1$, so D is closed disk $x^2 + y^2 \le 1$, and C_2 is a circular cylinder with radius 1 centered at the z axis. Let the bottom face of E be the hemisphere $z = -\sqrt{1 - x^2 - y^2}$. Let the top face of E be the hemishphere $z = 5 + \sqrt{1 - x^2 - y^2}$. So E is bounded by a section of C_2 having height 5, capped off above and below by hemispheres with radius 1. $\iiint_E f(x,y,z)\,dV =$

$$\iint_D \left[\int_{-\sqrt{1-x^2-y^2}}^{5+\sqrt{1-x^2-y^2}} f(x,y,z)\,dz \right] dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{5+\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$$

A special case of any of these triple integrals is where the the solid E has no lateral face, because the top surface and the bottom surface come together, forming a curve C_3 in x,y,z space–a curve whose orthogonal projection onto the x,y plane is the curve C_1 (assuming we have a Type 1 problem). A simple example of this would be if E is just a sphere. Think of the previous example, where the top and bottom hemispheres are just the two halves of one sphere, $x^2 + y^2 + (z-5)^2 = 1$, centered at the point $(0,0,5)$. The top hemisphere is $z = 5 + \sqrt{1 - x^2 - y^2}$, and the bottom hemisphere is $z = 5 - \sqrt{1 - x^2 - y^2}$. Thus,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{5-\sqrt{1-x^2-y^2}}^{5+\sqrt{1-x^2-y^2}} f(x,y,z)\,dz \right] dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{5-\sqrt{1-x^2-y^2}}^{5+\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$$

Example 2: Let E be the tetrahedron (triangular-based pyramid) bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$. The four vertices of this tetrahedron are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. We will approach this as a Type 1 problem, where D is the triangular region in the x,y plane with vertices $(0,0)$, $(1,0)$, and $(0,1)$, bounded by the lines $x = 0$, $y = 0$, and $y = -x + 1$. We will view D as Type I, with $g_1(x) = 0$ and $g_2(x) = -x + 1$. For E, we have $u_1(x,y) = 0$ and $u_2(x,y) = 1 - x - y$.

$$\iiint_E z\,dV = \int_0^1 \int_0^{-x+1} \int_0^{1-x-y} z\,dz\,dy\,dx.$$

$\displaystyle\int_0^{1-x-y} z\,dz = \left[\tfrac{1}{2}z^2 \right]_{z=0}^{z=1-x-y} = \tfrac{1}{2}(1-x-y)^2$. It’s better *not* to multiply this out.

Now we have $\displaystyle\int_0^1 \int_0^{-x+1} \tfrac{1}{2}(1-x-y)^2\, dy\, dx = \tfrac{1}{2}\int_0^1 \int_0^{-x+1} (1-x-y)^2\, dy\, dx.$

To find $\int (1-x-y)^2\, dy$, let $w = 1-x-y$, so $dw = -dy$ and $dy = -dw$, giving us $-\int w^2 dw = -\tfrac{1}{3}w^3 = -\tfrac{1}{3}(1-x-y)^3$. Hence, $\displaystyle\int_0^{-x+1} (1-x-y)^2\, dy =$ $\left[-\tfrac{1}{3}(1-x-y)^3\right]_{y=0}^{y=-x+1} = \tfrac{1}{3}\left[(1-x-y)^3\right]_{y=-x+1}^{y=0}$.

When $y = 0$, $(1-x-y)^3$ becomes $(1-x)^3$. When $y = -x+1$, $(1-x-y)^3$ becomes 0. Thus, $\tfrac{1}{3}\left[(1-x-y)^3\right]_{y=-x+1}^{y=0}$ gives us $\tfrac{1}{3}(1-x)^3$. Once again, it's better *not* to multiply this out.

Now we have $\displaystyle\tfrac{1}{2}\int_0^1 \tfrac{1}{3}(1-x)^3 dx = \tfrac{1}{6}\int_0^1 (1-x)^3 dx.$

To find $\int (1-x)^3 dx$, let $w = 1-x$, so $dw = -dx$ and $dx = -dw$, giving us $-\int w^3 dw = -\tfrac{1}{4}w^4 = -\tfrac{1}{4}(1-x)^4$. Hence,
$\displaystyle\tfrac{1}{6}\int_0^1 (1-x)^3 dx = \tfrac{1}{6}\left[-\tfrac{1}{4}(1-x)^4\right]_0^1 = \tfrac{1}{24}\left[(1-x)^4\right]_1^0 = \tfrac{1}{24}.$

Example 3: Let E be the region bounded by the circular paraboloid $y = x^2+z^2$ and the plane $y = 4$. We will approach this as a Type 3 solid. Its projection onto the x,z plane, D, is the circular disk $x^2+z^2 \le 4$. This is another case where E has no lateral face. Its bottom face is the surface $y = u_1(x,z) = x^2+z^2$, and its top face is the surface (or plane) $y = u_2(x,z) = 4$.

$$\iiint_E \sqrt{x^2+z^2}\; dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\right] dA =$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\, dz\, dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+z^2}\int_{x^2+z^2}^{4} dy\, dz\, dx.$$

$$\int_{x^2+z^2}^{4} dy = [y]_{y=x^2+z^2}^{y=4} = 4-(x^2+z^2) = 4-x^2-z^2.$$

Now we have $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+z^2}\,(4-x^2-z^2)\; dz\, dx.$

To complete the problem, it is best to switch to polar coordinates, with $x = r\cos\theta$ and $z = r\sin\theta$, and $x^2 + z^2 = r^2$. D is simply the polar rectangle $[0,2] \times [0,2\pi]$, so the double integral becomes $\int_0^{2\pi}\int_0^2 r(4 - r^2)\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^2 (4r^2 - r^4)\ dr\ d\theta = \int_0^2 (4r^2 - r^4)dr \int_0^{2\pi} d\theta$. Note that we have used the Integral Factorization Principle. We now get $\left[\frac{4}{3}r^3 + \frac{1}{5}r^5\right]_0^2 \bullet [\theta]_0^{2\pi} = (\frac{64}{15})(2\pi) = \frac{128\pi}{15}$.

Example 4: Rewrite $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\ dz\ dy\ dx$ in the order $dx\ dz\ dy$.

The integral starts out as Type 1, and we must rewrite it as Type 2. We first analyze it in the context of Type 1 in order to figure out the solid E.

E has lower boundary $z = 0$ and upper boundary $z = y$, which is an oblique plane (the orthogonal projection, parallel to the x axis, of the line $z = y$ in the y,z plane). Note that these lower and upper boundaries intersect at the x axis.

The region D in the x,y plane is Type I, where $x \in [0,1]$ and $y \in [0,x^2]$. In other words, the lower boundary of D is the x axis and the upper boundary is the parabola $y = x^2$, between the vertical lines $x = 0$ and $x = 1$. The corner points of D are $(0,0)$, $(1,0)$, and $(0,1)$.

In terms of our general theory, the boundary of D is a simple, closed curve C_1. In this case, C_1 consists of three distinct pieces: the line segment along the x axis from $(0,0)$ to $(1,0)$, the line segment along the vertical line $x = 1$ from $(1,0)$ to $(1,1)$, and the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$. Before trying to visualize region E, it's helpful to first visualize the cylinder C_2 of infinite extent. (E is derived from C_2 by capping it above and below.) Cylinder C_2 has three lateral faces: The section of the plane $y = 0$ where $x \in [0,1]$, the section of the plane $x = 1$ where $y \in [0,1]$, and the section of the parabolic cylinder $y = x^2$ where $x \in [0,1]$.

Since D lies in the x,y plane, which is the plane $z = 0$, and since $z = 0$ is the lower bounary of E, D is *itself* the bottom face of E. The top face intersects the bottom face at the x axis. Say we start out at the x axis, anywhere in the interval $[0,1]$. Say we hold x fixed and then allow y to increase from 0 to 1. As we move across the top face of E, i.e., across the plane $z = y$, z will likewise increase from 0 to 1 (at a 45 degree angle). For this fixed value of x, we reach the far edge of the top face when $z = y = x^2$. The maximum values of y and z are achieved when x is 1, and these maximum values are $z = y = 1$. Thus, the highest point of E is the point $(1,1,1)$. Every other point of E has a z coordinate less than 1.

Whereas cylinder C_2 has three lateral faces, E has only two: the section of the plane $x = 1$ where $y \in [0,1]$ and $z \in [0,y]$, and the section of the parabolic cylinder $y = x^2$ where $x \in [0,1]$ and $z \in [0,x^2]$. The latter face is difficult to visualize, but the former face is relatively straightforward: It is a triangular patch of the plane $x = 1$, with vertices $(1,0,0)$, $(1,1,0)$, and $(1,1,1)$.

To reformulate E as Type 2, we must find the projection of E onto the y,z plane. This would be the region D' bounded by the y axis, the z axis, and the line $z = y$. This is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

As a Type 2 solid, the upper boundary of E is the plane $x = 1$. The lower boundary is a bit trickier. In general, it is a surface of the form $x = u_1(y,z)$, but in this case u_1 depends only on y. To find this function, we invert the equation $y = x^2$, giving us $x = \sqrt{y}$.

Thus, we may rewrite our integral as $\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\ dx\ dz\ dy$.

Finding Volumes of Solids:

Scenario 1: The graph of a function $y = h(x)$ is a curve C that passes the vertical line test. If h is positive on its domain, then C lies above the x axis. For an interval $[a,b]$ in the domain of h, the (single) integral $\int_a^b h(x)\ dx$ gives us the area of the plane region bounded below by the x axis and bounded above by C over the interval $[a,b]$. If we replace $h(x)$ with 1, we get $\int_a^b dx$, which gives us the length of the interval $[a,b]$, which is $b - a$. (Technically, we should say that the *numerical value* of $\int_a^b dx$ is equal to the *numerical value* of the length of $[a,b]$. The *units*, however, are different. $\int_a^b dx$ will be in *square* units, whereas the length of $[a,b]$ will be in *linear* units.)

Scenario 2: The graph of a function $z = g(x,y)$ is a surface S that passes the vertical line test. If g is positive on its domain, then S lies above the x,y plane. For a closed, bounded region D in the domain of g, the double integral $\iint_D g(x,y)\ dA$ gives us the volume of the solid bounded below by the x,y plane and bounded above by S over the region D. If we replace $g(x,y)$ with 1, we get $\iint_D dA$, which gives us the area of D. (Technically, we should say that the *numerical value* of $\iint_D dA$ is equal to the *numerical value* of the area of D. The *units*, however, are different. $\iint_D dA$ will be in *cubic* units, whereas the area of D will be in *square* units.)

Now let us combine Scenarios 1 and 2. Suppose the region D is in fact the plane region bounded below by the x axis and bounded above by C over the interval $[a,b]$. Then D is a Type I region, with $g_1(x) = 0$ and $g_2(x) = h(x)$. The area of D can thus be expressed either as a single integral or as a double integral: $\int_a^b h(x)\ dx = \iint_D dA = \int_a^b \int_0^{h(x)} dy\ dx$.

Scenario 3: The graph of a function $w = f(x,y,z)$ is a hyper-surface H that passes the vertical line test. If f is positive on its domain, then H lies "above" x,y,z space. For a closed, bounded region E in the domain of f, the triple integral $\iiint_E f(x,y,z)\ dV$ gives us the hyper-volume of the hyper-solid bounded "below" by x,y,z space and bounded "above" by H over the solid E. If we replace $f(x,y,z)$ with 1, we get $\iiint_E dV$, which gives us the volume of E. (Technically, we should say that the *numerical value* of $\iiint_E dV$ is equal to the *numerical* *value* of the volume of E. The *units*, however, are different. $\iiint_E dV$ will be in *quartic* units, whereas the volume of E will be in *cubic* units.)

Now let us combine Scenarios 2 and 3. Suppose the region E is in fact the solid bounded below by the x,y plane and bounded above by S over the region D. Then E is a Type 1 solid, with $u_1(x,y) = 0$ and $u_2(x,y) = g(x,y)$. The volume of E can thus be expressed either as a double integral or as a triple integral: $\iint_D g(x,y)\, dA = \iiint_E dV = \iint_D \int_0^{g(x,y)} dz\, dA$.

Of course, we may be able to combine all three scenarios, giving us $\int_a^b \int_0^{h(x)} \int_0^{g(x,y)} dz\, dy\, dx$. But that's not the point of this analysis. The point is that we have the flexibility to calculate areas using either single or double integrals (in the latter case, with an integrand of 1), and we have the flexibility to calculate volumes using either double or triple integrals (in the latter case, with an integrand of 1).