15.9) Integral Transformations

Preliminary Discussion:

Say we have a function, $u = 2x + 6.$ We say u is a *function of* $x,$ or u *depends on* $x.$ We refer to u as the *dependent variable* and to x as the *independent variable*.

We may "invert" the equation by solving for x in terms of $u,$ giving us a new function, $x = \frac{1}{2}u - 3.$ Now x is a function of $u,$ or x depends on $u.$ We now have dependent variable x and independent variable $u.$

When we graph a function, we place the independent variable on the horizontal axis and the dependent variable on the vertical axis. Thus, when we graph the function $u = 2x + 6,$ we place x on the horizontal axis and u on the vertical axis. We refer to this setup as "the x, u plane." On the other hand, when we graph the function $x = \frac{1}{2}u - 3,$ we place u on the horizontal axis and x on the vertical axis. We refer to this setup as "the u,x plane."

In the x, u plane, the graph of the function $u = 2x + 6$ is a line with slope 2 and u intercept $(0, 6).$

In the u,x plane, the graph of the function $x = \frac{1}{2}u - 3$ is a line with slope $\frac{1}{2}$ and x intercept $(0, -3).$

These two functions are said to be *inverses* of each other. If you choose a particular value of x and plug it into the equation $u = 2x + 6,$ you get a particular value of $u.$ If you take that value of u and plug it into the equation $x = \frac{1}{2}u - 3,$ you get back your *original* value of $x.$ It also works the other way around. For instance, if $x = 5,$ then $u = 16,$ and if $u = 16,$ then $x = 5.$ Similarly, if $u = 4,$ then $x = -1,$ and if $x = -1,$ then $u = 4.$ (We can say that each function "undoes" the other.)

If we adopt traditional function notation, as learned in College Algebra, we may refer to the first function as, say, $f,$ and we may refer to the second function as, say, $g.$ Then we may write $u = f(x) = 2x + 6$ and $x = g(u) = \frac{1}{2}u - 3.$ The functions f and g are *inverse functions* with respect to each other. The inverse of f is denoted f^{-1} (pronounced "f inverse"), and the inverse of g is denoted g^{-1} (pronounced "g inverse"). Thus, $f^{-1} = g$ and $g^{-1} = f.$

Instead of thinking in terms of functions, we can adopt a new framework, the framework of *transformations* between two different *spaces*.

Let T be a transformation *from* the u axis, or "u space," *to* the x axis, or "x space,", defined by the equation $x = \frac{1}{2}u - 3.$ The *reverse transformation*, denoted $T^{-1},$ is *from* the x axis, or "x space," *to* the u axis, or "u space,", defined by the equation $u = 2x + 6.$

T^{-1} is pronounced "T inverse," but you could also read it as "inverse $T,$" or "reverse $T.$"

You might wonder, why do we call the former transformation T and the latter transformation T^{-1}, rather than the other way around? It doesn't really matter, since it's just semantics, but the way I've named them is the usual way of doing it.

Transformations Between Two-Dimensional Spaces:

The above example illustrated transformations between *one-dimensional* spaces (with each space represented by a single variable). Now let us consider transformations between *two-dimensional* spaces (with each space represented by two variables).

Let T be a transformation *from* the u,v plane (where the horizontal axis is the u axis and the vertical axis is the v axis), or "u,v space," *to* the x,y plane (where the horizontal axis is the x axis and the vertical axis is the y axis), or "x,y space," defined by the following pair of equations:

- $x = 7u + 5v$
- $y = 4u + 3v$

The reverse transformation is T^{-1}, which is *from* the x,y plane, or "x,y space," *to* the u,v plane, or "u,v space," defined by the following pair of equations:

- $u = 3x - 5y$
- $v = -4x + 7y$

We may write $T(u,v) = (x,y)$ and $T^{-1}(x,y) = (u,v)$.

To illustrate: $T(6,-2) = (32,18)$, and $T^{-1}(11,8) = (-7,12)$.

We say that the point $(32,18)$ in x,y space is the *image*, under the transformation T, of the point $(6,-2)$ in u,v space. For brevity, we could say $(32,18)$ is the T image of $(6,-2)$.

Likewise, we say that the point $(-7,12)$ in u,v space is the *image*, under the transformation T^{-1}, of the point $(11,8)$ in x,y space. For brevity, we could say $(-7,12)$ is the the T^{-1} image of $(11,8)$.

Each transformation "undoes" the other. Thus, $T^{-1}(32,18) = (6,-2)$, while $T(-7,12) = (11,8)$. In other words, $T^{-1}(T(u,v)) = (u,v)$ and $T(T^{-1}(x,y)) = (x,y)$. This is true in general, provided the following conditions hold:

- Both transformations are **one-to-one**. T is one-to-one if no two points in u,v space map to the same point in x,y space. (In other words, two different points in u,v space must have two different images in x,y space.) T^{-1} is one-to-one if no two points in x,y space map to the same point in u,v space. (In other words, two different points in x,y space must have two different images in u,v space.)
- T maps the entire u,v plane onto the entire x,y plane, while T^{-1} maps the entire x,y plane onto the entire u,v plane.

Given the equations for $T,$ we may find the equations for T^{-1} by *solving* the given equations *for* u and v *in terms of* x and $y.$ But this means solving the given equations *simultaneously*, as a *system* of equations, using the methods of College Algebra. In the above example, we have a *linear* system (i.e., a system of linear equations), so we may use either Elimination by Addition or Substitution. Here's how it would work, using the former method...

Given the system $7u + 5v = x, \;\; 4u + 3v = y,$ let's start by eliminating $u.$ Multiply the first equation by 4 and the second by $-7.$ We get $28u + 20v = 4x$ and $-28u - 21v = -7y.$ Adding these gives us $-v = 4x - 7y,$ or $v = -4x + 7y.$

Now let's eliminate $v.$ Returning to the original pair of equations, multiply the first by 3 and the second by $-5.$ We get $21u + 15v = 3x$ and $-20u - 15v = -5y.$ Adding these gives us $u = 3x - 5y.$

Thus we obtain the equations for T^{-1}, namely, $u = 3x - 5y$ and $v = -4x + 7y.$ The same technique could be used if we *started* with the equations for T^{-1} and had to find the equations for $T.$

Previously, when we calculated $T(6, -2)$ and $T^{-1}(11, 8),$ we were dealing with transformations of *points*. The transformation of a point in one space is a point in the other space. Now let us consider transformations of *functions* or *relations*. The transformation of a function or relation in one space is a function or relation in the other space. When dealing with *functions*, we adopt the following convention: In x, y space, x is the independent variable and y is the dependent variable. In u, v space, u is the independent variable and v is the dependent variable. (If a relation is *not* a function, then we do not designate the variables as independent and dependent.)

Say we start with the function $y = 2x + 1$ in x, y space. Let us find the corresponding function in u, v space, using the transformations defined above. Since we are going from x, y space to u, v space, we would have to say, technically, that the new function is the image of the original function under T^{-1}. However, as you are about to see, we will actually use the formulas from T to find the desired function.

Before proceeding, here's a useful observation. Theoretically, the function $y = 2x + 1$ is a set of points that passes the vertical line test in x, y space. Each point individually can be transformed, under $T^{-1},$ to a corresponding point in u, v space. What we are doing is transforming the entire *set* of points, which gives us a *set* of corresponding points in u, v space. Generally speaking, this set of points will be at least a relation in u, v space; it may or may not be a function, depending on whether it passes the vertical line test in u, v space. In this particular example, we *are* indeed going to get a function, as you will see in a moment. At any rate, to find the new function or relation, we do not transform points one at a time; instead, we manipulate the equation for the original function, using algebraic substitutions. Let's see how this works in the present example...

Given the equation $y = 2x + 1$, we refer to the formulas from T, so we subtitute $7u + 5v$ in place of x, and we substitute $4u + 3v$ in place of y. We obtain the equation $4u + 3v = 2(7u + 5v) + 1$. Solving for v, be get $v = \frac{-10}{7}u - \frac{1}{7}$. This function is the T^{-1} image of $y = 2x + 1$.

We may write $T^{-1}\{y = 2x + 1\} = \{v = \frac{-10}{7}u - \frac{1}{7}\}$. This is the casual way of writing it. If we were going to be formal, we would write $T^{-1}\{(x,y) \mid y = 2x + 1\} = \{(u,v) \mid v = \frac{-10}{7}u - \frac{1}{7}\}$.

Just to check that our answer makes sense, consider this. A point on the graph of $y = 2x + 1$ is $(3,7)$. The T^{-1} image of $(3,7)$ is $(-26,37)$. This latter point should lie on the graph of the function $v = \frac{-10}{7}u - \frac{1}{7}$. Does it? Well, when u is -26, $v = \frac{-10}{7}(-26) - \frac{1}{7} = \frac{260}{7} - \frac{1}{7} = \frac{259}{7} = 37$, so yes, it does!

As another example, consider the relation $x^2 + y^2 = 1$. The T^{-1} image of this relation is $(7u + 5v)^2 + (4u + 3v)^2 = 1$, or $65u^2 + 94uv + 34v^2 = 1$. This is an ellipse in u,v space, but it is an unusual ellipse, because its two axes are oblique (i.e., slanted). (Don't worry about that!!)

We may write $T^{-1}\{x^2 + y^2 = 1\} = \{65u^2 + 94uv + 34v^2 = 1\}$.

Now consider the function $v = u^2$, whose graph is a parabola. The T image of this function is found by using the formulas of T^{-1}. In other words, we substitue $3x - 5y$ in place of u and $-4x + 7y$ in place of v. We obtain $-4x + 7y = (3x - 5y)^2$, or $9x^2 - 30xy + 25y^2 + 4x - 7y = 0$. This is a parabola in x,y space, but it is an unusual parabola, because its axis of symmetry is oblique. (Again, don't worry about that!!) By the way, this slanted parabola fails the vertical line test, since it has two y intercepts, $(0,0)$ and $(0,\frac{7}{25})$, so this relation is not a function. This shows that the image of a function may not be a function.

We may write $T\{v = u^2\} = \{9x^2 - 30xy + 25y^2 + 4x - 7y = 0\}$.

Transformations of Regions:

So far, we have considered transformations of *points* and transformations of *functions* or *relations*. Now let us consider transformations of *regions*. We generally refer to a region in x,y space as R and to the corresponding region in u,v space as S.

Let T be the transformation defined by the equations $x = u + v$, $y = u - v$. The reverse transformation is T^{-1}, defined by the equations $u = \frac{1}{2}x + \frac{1}{2}y$, $v = \frac{1}{2}x - \frac{1}{2}y$.

Let R be the closed triangular region in the x,y plane bounded by the lines $y = x$, $y = -x + 2$, and $y = 0$. The vertices of this triangle are the points $A = (0,0)$, $B = (1,1)$, and $C = (2,0)$. A is the intersection of the first and third lines, B is the intersection of the first and second lines, and C is the intersection of the second and third lines. Region R is the triangle ABC along with its interior.

The T^{-1} image of region R will be a region S in the u,v plane. Let's find this region.

$T^{-1}\{y = x\} = \{v = 0\}$, which is the u axis.

$T^{-1}\{y = -x + 2\} = \{u = 1\}$, which is a vertical line.

$T^{-1}\{y = 0\} = \{v = u\}$, which is a line through the origin with slope 1.

These three lines bound a triangle in the u,v plane, with vertices $(0,0)$, $(1,1)$, and $(1,0)$. These three vertices are the T^{-1} images of the vertices of triangle ABC. Specifically, $T^{-1}(0,0) = (0,0)$, $T^{-1}(1,1) = (1,0)$, and $T^{-1}(2,0) = (1,1)$. Hence, we shall name the vertices of our new triangle, in u,v space, as follows: $(0,0) = A'$, $(1,0) = B'$, $(1,1) = C'$. Our new triangle may be named $A'B'C'$.

Bear in mind, region R is not just triangle ABC, it is that triangle together with its interior. Likewise, region S is not just triangle $A'B'C'$, it is that triangle together with its interior. Points on triangle ABC are *boundary points* of R, and points on triangle $A'B'C'$ are *boundary points* of S. Boundary points of R correspond to boundary points of S, while interior points of R correspond to interior points of S. For instance, the point $(1, \frac{1}{2})$ in x,y space is an interior point of R. Its image is the point $(\frac{3}{4}, \frac{1}{4})$ in u,v space, which is an interior point of S.

Notice that triangles ABC and $A'B'C'$ are both right triangles. The legs (i.e., the perpendicular sides) of the former triangle are $\overline{AB}$ and $\overline{BC}$. The legs of the latter triangle are $\overline{A'B'}$ and $\overline{B'C'}$. But there is an important difference. The legs of the former triangle are oblique, whereas the legs of the latter triangle are horizontal and vertical. If we were going to work out a double integral over each region, it would be easier to integrate over S than it would be to integrate over R. For instance, to integrate $u^2 + v^3$ over S, we could integrate as follows: $\iint_S (u^2 + v^3)dA = \int_0^1 \int_0^u (u^2 + v^3)dv\, du.$

So far, we have considered transformations between two-dimensional spaces, x,y space and u,v space. We typically think of these spaces as being *two distinct planes*. However, it is also possible that the spaces could be the *same plane* coordinatized in *two different coordinate systems*–the Cartesian Coordinate System and the Polar Coordinate System. We refer to the former as x,y space and the latter as r,θ space. In this situation, a region R in x,y space and the corresponding region S in r,θ space would be the exact same set of points, but the equations or inequalities representing the region in x,y space would be different from the equations or inequalities representing the region in r,θ space. For example, the closed disk of radius 5 centered at the origin (or pole) would be represented by the inequality $x^2 + y^2 \leq 25$ in x,y space and by the inequality $r \leq 5$ in r,θ space.

Transformations of Double Integrals:

Let T be the transformation defined by the equations $x = 2u,\ \ y = 4u + v$. The reverse transformation is T^{-1}, defined by the equations $u = \frac{1}{2}x,\ \ v = -2x + y$. Let R be the closed region in x,y space bounded by the parallelogram with vertices $(0,0)$, $(0,1)$, $(2,5)$, and $(2,4)$. In other words, R is the closed region bounded by the lines $x = 0,\ x = 2,\ y = 2x$, and $y = 2x + 1$. The image of R is the region S in u,v space, which is the closed region bounded by the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. In other words, S is the closed region bounded by the lines $u = 0,\ u = 1,\ v = 0$, and $v = 1$.

Say we are asked to evaluate the double integral $\iint_R \sqrt{2xy - 4x^2}\,dA$. This problem is challenging, for two reasons. First, the region R is not a rectangle; we know how to deal with it, but it's more complicated than dealing with a rectangle. Second (and more importantly), integrating $\sqrt{2xy - 4x^2}$ would be daunting.

To simplify the problem, we introduce an **integral transformation**. In other words, we find a transformation from x,y space into another space, such that the given integral in x,y space is transformed into a new integral, involving different variables, where the region is easier to deal with and (more importantly) the integrand becomes easier to integrate.

Guess what? The transformation defined three paragraphs ago is just what the doctor ordered! We have already seen that under this transformation, the parallelogram region R transforms to a rectangular region S. What about the integrand? To rewrite it, we use the formulas from T. Substituting $2u$ in place of x and $4u + v$ in place of y, $\sqrt{2xy - 4x^2}$ becomes $\sqrt{2(2u)(4u+v) - 4(2u)^2}$, which simplifies to $2\sqrt{uv}$, or $2u^{1/2}v^{1/2}$.

In order for the transformed integral to be equivalent to the original, we must introduce into the new integrand the **Jacobian Factor**, $|J|$, which is the absolute value of the **Jacobian Determinant**, $J(u,v)$, which is the determinant of the **Jacobian Matrix**, which is $\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$. In other words, $J(u,v) = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$. Thus, to find the Jacobian Factor, first set up the Jacobian Matrix, then calculate its determinant, which is the Jacobian Determinant, then take the absolute value of that determinant.

In this example, the Jacobian Matrix is $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$. The Jacobian Determinant is $J(u,v) = 2$, so the Jacobian Factor is 2.

Thus, $\iint_R \sqrt{2xy - 4x^2}\,dA = \iint_S (2u^{1/2}v^{1/2})(2)dA = 4\iint_S u^{1/2}v^{1/2}dA =$

$$4\int_0^1\int_0^1 u^{1/2}v^{1/2}du\,dv = 4\int_0^1 v^{1/2}dv\int_0^1 u^{1/2}du = 4\left[\tfrac{2}{3}v^{3/2}\right]_0^1\left[\tfrac{2}{3}u^{3/2}\right]_0^1 = 4(\tfrac{2}{3})(\tfrac{2}{3}) = \tfrac{16}{9}.$$

In Section 15.3, we learned how to rewrite a double integral using polar coordinates. This can now be seen as an integral transformation. Let T be a transformation from r,θ space to x,y space, defined by the equations $x = r\cos\theta,\ \ y = r\sin\theta$. The Jacobian Matrix is $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$. Since $x = r\cos\theta$ and $y = r\sin\theta$, we get $\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$. The determinant of this matrix is $r\cos^2\theta + r\sin^2\theta = r$. Since r is nonnegative, $|r| = r$. Thus, our Jacobian Factor is r.

Transformations of Triple Integrals:

Above, we evaluated a difficult double integral by means of an integral transformation. This strategy can also be applied to triple integrals. Generally speaking, we find a transformation between x,y,z space and u,v,w space, so that a region R in the former space maps to a (presumably simpler) region S in the latter space, and so that the integrand in the former space transforms to a (presumably simpler) integrand in the latter space. As before, in order for the two integrals to be equivalent, we must introduce into the new integrand the Jacobian Factor, $|J|$, which is again the absolute value of the Jacobian Determinant, which is now $J(u,v,w)$. This is the determinant of the Jacobian Matrix, which is now

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

In the preceding paragraph, I referred to the new space as u,v,w space. Those are the variables we typically use if the new space is Cartesian (i.e., if it uses a rectangular coordinate system). On the other hand, if the new space uses cylindrical coordinates, we would write r,θ,z in place of u,v,w. In this case, the Jacobian Matrix would be $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix}$, which simplifies to $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & 0 \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since $x = r\cos\theta$ and $y = r\sin\theta$, we get $\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The determinant of this matrix is r (use cofactor expansion along the third row). Since r is nonnegative, $|r| = r$. Thus, our Jacobian Factor is r.

If the new space uses spherical coordinates, we would write ρ, φ, θ in place of u, v, w. In this case, the Jacobian Matrix would be $\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{bmatrix}$, which is $\begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix}$. The determinant of this matrix is $\rho^2\sin\varphi$ (use cofactor expansion along the third row). The spherical coordinate system has been defined so that $\varphi \in [0, \pi]$, and therefore $\sin\varphi \geq 0$. Consequently, $\rho^2\sin\varphi \geq 0$, which means $|\rho^2\sin\varphi| = \rho^2\sin\varphi$. Thus, our Jacobian Factor is $\rho^2\sin\varphi$.

Summary:

- When transforming from rectangular x, y space to polar r, θ space, $|J| = r$.
- When transforming from rectangular x, y, z space to cylindrical r, θ, z space, $|J| = r$.
- When transforming from rectangular x, y, z space to spherical ρ, φ, θ space, $|J| = \rho^2\sin\varphi$.

Addendum:

Consider the following problem from Calculus I.

Find $\int_0^1 \sqrt{2x+6}\,dx$.

We use the basic integration technique of Substitution.

Let $u = 2x + 6$. So $du = 2\,dx$, and $\frac{1}{2}du = dx$.

When $x = 0$, $u = 6$, and when $x = 1$, $u = 8$.

Thus, $\int_0^1 \sqrt{2x+6}\,dx = \int_6^8 \sqrt{u}\,\frac{1}{2}du = \frac{1}{2}\int_6^8 u^{1/2}du = \frac{1}{2}\frac{2}{3}[u^{3/2}]_6^8 = \frac{1}{3}[u^{3/2}]_6^8 = \frac{1}{3}(8^{3/2} - 6^{3/2}) \approx 2.643.$

We can think of this an an integral transformation, based on the transformation introduced in our Preliminary Discussion:

Let T be the transformation from u space to x space defined by the equation $x = \frac{1}{2}u - 3$. The reverse transformation, T^{-1}, from x space to u space, is defined by the equation $u = 2x + 6$.

In this case, since there is only one variable in each space, we use ordinary derivatives instead of partial derivatives. The Jacobian Matrix is now a one-by-one matrix (i.e., one row and one column, hence only one entry), namely, $\left[\frac{dx}{du}\right]$. The determinant of any one-by-one matrix is simply its sole element. Hence, the Jacobian Determinant is just $\frac{dx}{du}$, and so the Jacobian Factor is $\left|\frac{dx}{du}\right|$, i.e., the absolute value of $\frac{dx}{du}$.

Here, $\frac{dx}{du} = \frac{1}{2}$, so $|J| = \frac{1}{2}$.

In this problem, R is the interval $[0,1]$ on the x axis. Its image under T^{-1}, which we call S, is the interval $[6,8]$ on the u axis. Note that $T^{-1}(0) = 6$ and $T^{-1}(1) = 8$. For every point on the x axis between 0 and 1, $T^{-1}(x)$ is a point between 6 and 8 on the u axis; for instance, $T^{-1}(0.5) = 7$.

Since $x = \frac{1}{2}u - 3$, we could rewrite our original integrand, $\sqrt{2x+6}$, by substituting $\frac{1}{2}u - 3$ in place of x. That would give us $\sqrt{2(\frac{1}{2}u-3)+6} = \sqrt{u-6+6} = \sqrt{u}$. However, it's easier just to observe that $u = 2x + 6$, so $\sqrt{2x+6} = \sqrt{u}$.

Hence, in Calculus I, you were actually doing integral transformations and dealing with Jacobian Factors, all without realizing it!

Consider one more example:

Find $\int_0^3 x\sqrt{9-x^2}\,dx$.

Here's the way we'd do it in Calculus I...

First, rewrite $x\sqrt{9-x^2}\,dx$ as $\sqrt{9-x^2}\ x\,dx$.

Let $u = 9 - x^2$. So $du = -2x\,dx$, and $-\frac{1}{2}du = x\,dx$.

When $x = 0$, $u = 9$, and when $x = 3$, $u = 0$.

Thus, $\int_0^3 \sqrt{9-x^2}\ x$

$dx = \int_9^0 \sqrt{u}\,(-\frac{1}{2})du = -\frac{1}{2}\int_9^0 \sqrt{u}\,du = \frac{1}{2}\int_0^9 \sqrt{u}\,du = \frac{1}{2}\,\frac{2}{3}[u^{3/2}]_0^9 = \frac{1}{3}[u^{3/2}]_0^9 = \frac{1}{3}(27) = 9.$

If we wanted to do it the Calculus III way, it'd look like this...

Let T be the transformation from u space to x space defined by the equation $x = \sqrt{9-u}$. The reverse transformation, T^{-1}, from x space to u space, is defined by the equation $u = 9 - x^2$.

$\frac{dx}{du} = \frac{-1}{2\sqrt{9-u}}$, so the Jacobian Matrix is $\left[\frac{dx}{du}\right] = \left[\frac{-1}{2\sqrt{9-u}}\right]$ and the the Jacobian Determinant is $\frac{-1}{2\sqrt{9-u}}$. Hence the Jacobian Factor is $\left|\frac{-1}{2\sqrt{9-u}}\right| = \frac{1}{2\sqrt{9-u}}$.

R is the interval $[0,3]$ on the x axis. Its image under T^{-1}, which we call S, is the interval $[0,9]$ on the u axis. Note that $T^{-1}(0) = 9$ and $T^{-1}(3) = 0$. For every point on the x axis between 0 and 3, $T^{-1}(x)$ is a point between 0 and 9 on the u axis; for instance, $T^{-1}(2) = 5$.

Now let's rewrite the integrand. Substituting $\sqrt{9-u}$ in place of x and u in place of $9 - x^2$, $x\sqrt{9-x^2}$ becomes $\sqrt{9-u}\,\sqrt{u}$. We could multiply these into a single radical, but it's better to leave it in this form. Why? Because when we multiply the Jacobian Factor into the integrand, $\sqrt{9-u}$ will cancel out, leaving us with $\frac{1}{2}\sqrt{u}$. Of course, the $\frac{1}{2}$ will factor out of the integral.

Thus, $\int_0^3 x\sqrt{9-x^2}\,dx = \frac{1}{2}\int_0^9 \sqrt{u}\,du$, which we evaluate as before, giving us 9.

In this Addendum, we have seen how Calculus I problems solved via Substitution could be viewed as integral transformations. It isn't *necessary* to look at such problems this way. If you have a Calculus I level problem, feel free to solve it by the methods of Calculus I. (I just wanted to show you that you *could* look at such problems as integral transformations.)

In fact, you might say that Substitution in Calculus I is a special case of the Calculus III method of integral transformation. Namely, it's the case where we have a one-dimensional domain (in other words, where we are working with a single integral as opposed to a multiple integral).