2.2 The Limit of a Function

In this section we will discuss the definition of a limit and analyze how limit arise when using numerical and graphical methods to compute them.

Intuitive Definition of a Limit: Suppose f(x) is defined when x I near the number a.
Then we write

$$\lim_{x \to a} f(x) = L$$

and we say "the limit of f(x) as x approaches a equals L"

We can make the values of f(x) arbitrarily close to L (as close to L as we would like) by restricting x to be sufficiently close to **a** (on either side of **a** but not equal to **a**).

In other words, the value of f(x) tends to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but x ≠ a.

Sometimes this is also written as $\boldsymbol{f(x) \to L \text{ as } x \to a}$

Note: The phrase "but x ≠ a" in the definition of the limit. This means that in finding the limit of f(x) as x approaches a, we never consider x = a. As a matter of fact, f(x) need not even be defined when x = a. The only thing that matters is how f is defined near a.

Example: Find the value of $\boldsymbol{\lim_{x \to 3} \frac{x-3}{x^2-9}}$. Notice that the function f(x) = $\frac{x-3}{x^2-9}$ is not defined when x = 3, but that doesn't matter because the definition of a limit only considers values of x that are close to **a** but not equal to **a**. Because of this reason we can analyze what is happening to the left and right of $x = 3$ with a table.

$x < 3$	f(x)
2.5	0.18182
2.75	0.17391
2.9	0.16949
2.99	0.16694
2.999	0.16669
2.9999	0.16667

$x > 3$	f(x)
3.5	0.15385
3.25	0.16000
3.1	0.16393
3.01	0.16639
3.001	0.16664
3.0001	0.16666

(Notice that x is getting very, very close to 3 without actually equaling 3.)

Based on the tables above, we can guess that $\boldsymbol{\lim_{x \to 3} \frac{x-3}{x^2-9}}$ would be somewhere between 0.16667 and 0.16666. In fact, if you continue choosing values of x to get closer and closer to 3, then f(x) gets more 6s in the decimal (If x= 2.999999999⋯, then f(x) = 1.666666666⋯, which equals $\frac{1}{6}$. So we can say

$$\lim_{x\to 3}\frac{x-3}{x^2-9}=\frac{1}{6}$$

Now let's consider a more complicated example.

Example: Find the value of $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ Notice that the function f(x) is not defined at x = 0 (it actually can't be simplified to one which is defined at x = 0), but this doesn't matter because the definition of $\lim_{x\to a} f(x)$ says that we can consider values of x that are close to but not equal to a. Again, we need to make a set of tables to analyze what is happening to the value of the function as we approach x = 0 from both the left and right sides of zero.

$x<0$	$f(x)$
-1	0.8647
$-.5$	1.2642
$-.1$	1.8127
$-.01$	1.9801
$-.001$	1.9980
$-.0001$	1.9998

$x>0$	$f(x)$
1	6.3891
.5	3.4366
.1	2.2140
.01	2.0201
.001	2.0020
.0001	2.0002

Based on the tables, we can guess that $\lim_{x\to 0}\frac{e^{2x}-1}{x}=2$

Although the tables suggest that the limit of the first example is 0.1667, it by no means establishes that fact conclusively. Using a table can only suggest a value for the limit. Let's look at another example.

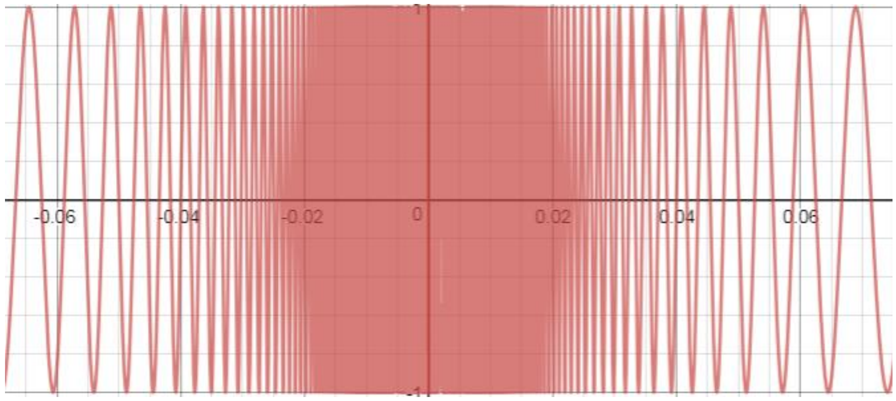
Example: Find the value of $\lim_{x\to 0} sin\left(\frac{\pi}{x}\right)$ Notice that the function $f(x)=sin\left(\frac{\pi}{x}\right)$ is undefined at x = 0.

Let's make tables approaching x = 0 from the left and the right.

$x<0$	$f(x)$
-1	0
$-.5$	0
$-.1$	0
$-.01$	0
$-.001$	0

$x>0$	$f(x)$
1	0
.5	0
.1	0
.01	0
.001	0

We might guess that the $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = 0$ but this is not true. Upon closer examination of this function we can see that the value of f(x) is oscillating between -1 and 1. The closer x gets to 0, the faster the function oscillates. No matter how close we get to zero, the function will continue to oscillate between -1 and 1 therefore a limit of this function as x approaches zero does not approach a single value, so the limit does not exist.



Now, let's consider one-sided limits.

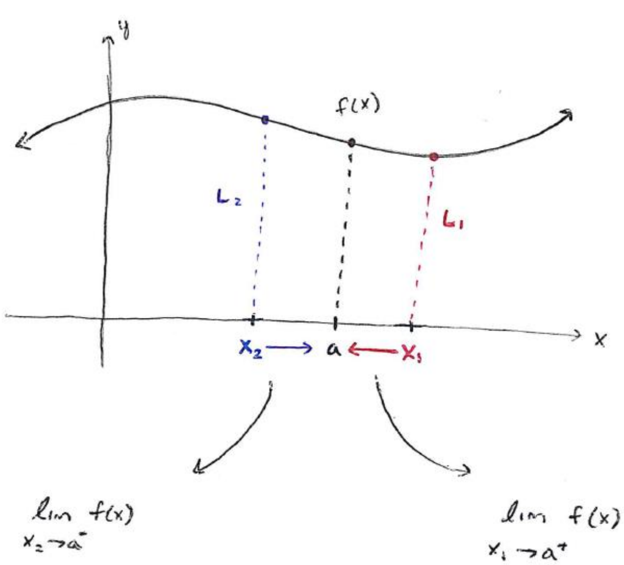
Definition of One-sided Limits:

We write $\lim_{x \to a^-} f(x) = L$ and say that the "left-hand" or "left-sided" limit of f(x) as x approaches **a** from the left side (or negative side) of **a** is equal to L. We can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to a with x **less than** a.

Similarly, if we require that x be greater than a, we get the "right-hand" or "right-sided" limit of f(x) as x approaches **a** from the right side (or positive side) of **a**, f(x) approaches L. We write this as

$$\lim_{x \to a^+} f(x) = L$$

Below is a graphical view of what the definition means.



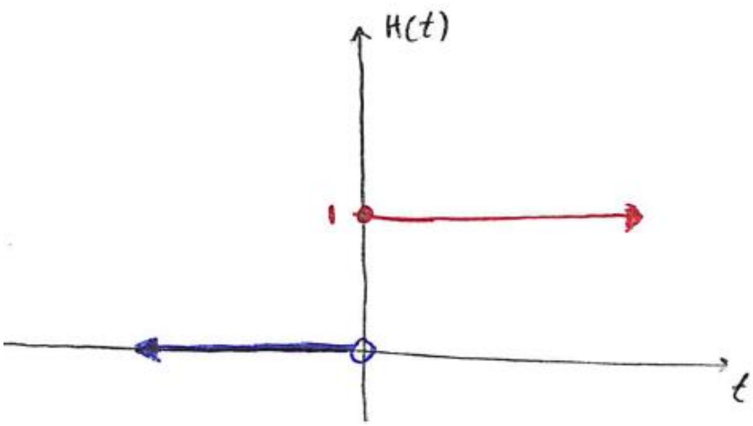
Now by comparing the first definition given in this section with thee two 0ne-sided limit definitions, we get the following:

$$\lim_{x\to a} f(x) = L \;\; if\ and\ only\ if \;\; \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Notice this is exactly what we did in the previous examples by making tables showing that **x** approached **a** from the right and left sides of a.

Example: The Heaviside Function H is defined by $H(t) = \begin{Bmatrix} 0 & if\ t < 0 \\ 1 & if\ t \geq 0 \end{Bmatrix}$. Find $\lim_{x\to 0} H(t)$.

First let's plot a graph of the function. Notice that this is a piecewise function.



From the graph we can see that as t approaches 0 from the left, the value of H(t) approaches 0. But as t approaches 0 from the right, the value of H(t) approaches 1. Mathematically we get:

$$\lim_{t\to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t\to 0^+} H(t) = 1$$

There is no single number that H(t) approaches as t approaches 0, therefore; $\lim_{t\to 0} H(t)$ **does not exist**. (DNE)

Example: Pg 92 #4

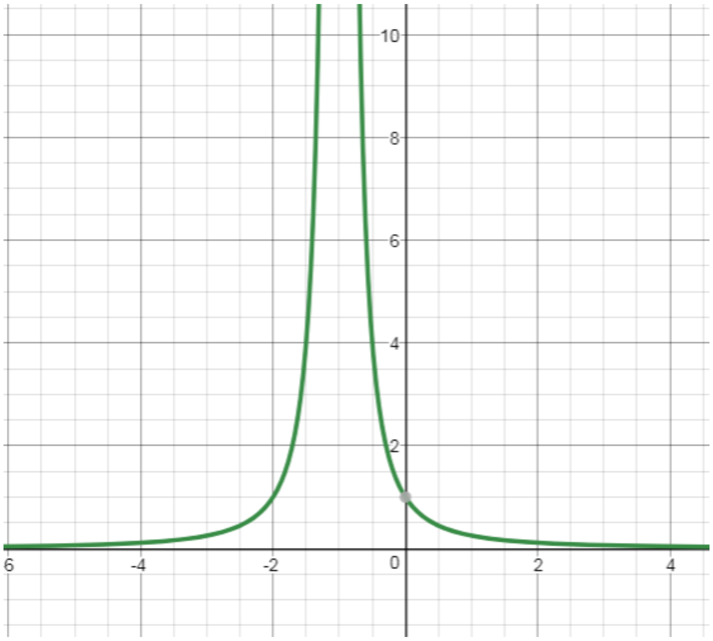
a.) $\lim_{x\to 2^-} f(x) = 3$ b.) $\lim_{x\to 2^+} f(x) = 1$ c.) $\lim_{x\to 2} f(x) = DNE$

d.) $f(2) = 3$ e.) $\lim_{x\to 4} f(x) = 4$ f.) $f(4) = DNE$

Infinite Limits

Now we will analyze functions that increase without bound as **x** approaches a specific value, **a**.

Example: Find $\lim_{x\to .1} \frac{1}{(x+1)^2}$ if it exists. Here is a graph of the function.



Notice that this function has a domain of $(-\infty,-1)\cup(-1,\infty)$, which means that this function is not defined at x = -1.

Let's make a table to see what happens as we approach x = -1 from the left and right.

$x < -1$	f(x)
-2	1
-1.5	4
-1.1	100
-1.01	10,000
-1.001	1,000,000

$x > -1$	f(x)
0	1
$-.5$	4
$-.9$	100
$-.99$	10,000
$-.999$	1,000,000

Notice that as we approach x = -1 from both sides, the values of y get very large but do not approach a specific value. For this reason we say that the limit does not exist.

$\lim_{x\to-1}\frac{1}{(x+1)^2}$ ***Does Not Exist*** this type of behavior is written as $\mathbf{\lim_{x\to-1}\frac{1}{(x+1)^2}} = \infty$

It is very important to understand that this does not mean that the limit exists or that ∞ is somehow a definite number, it is the way we express that the limit does not exist and that the function approaches infinity as x approaches a specific value.

Intuitive Definition of an Infinite Limit: Let f be a function defined on both sides of a, except possibly at a itself. Then –

$$\mathbf{\lim_{x\to a} f(x) = \infty}$$

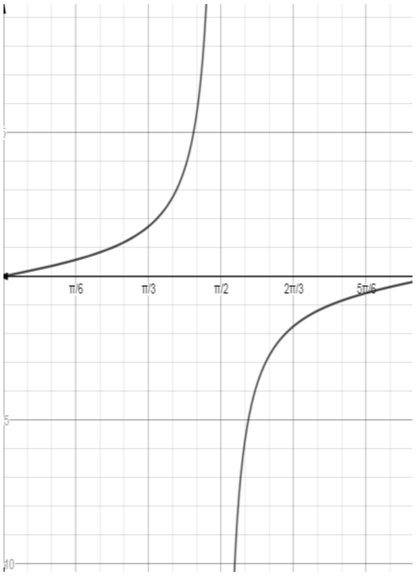
This means that values of f(x0 can be made arbitrarily large (as large as we please) by taking x sufficiently close to **a**, but not equal to a.

Definition: Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

This means that the values of f(x) can be made arbitrarily small (negatively large) by taking x sufficiently close to **a**, but not equal to a.

Example: Find $\lim_{x\to\frac{\pi}{2}} \tan(x)$ *if it exists.* The graph of f(x) = tan(x) $[0, \pi]$ is below.



Notice that $\lim_{x\to\frac{\pi}{2}^-} \tan(x) = \infty$ and $\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty$. Therefore the $\lim_{x\to\frac{\pi}{2}} \tan(x)$ does not exist.

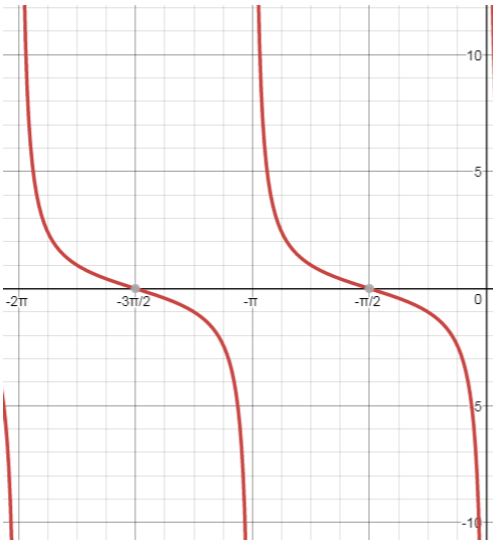
Furthermore, this example gives us a way to interpret vertical asymptotes using limits.

Definition: The vertical line "*x = a*" is called a vertical asymptote of the curve ***y=f(x)*** if at least one of the following statements is true.

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

Example: Find $\lim_{x\to\pi^-} \cot(x)$ The graph of f(x) = cot(x) $[-2\pi, 0]$ is shown below.



Notice that as x approaches $-\pi$ from the left side, cot(x) approaches $-\infty$, therefore; $\lim_{x\to-\pi} \cot(x) = -\infty$. The line $x = -\pi$ is a vertical asymptote of $f(x) = \cot(x)$.