## 2.4 The Precise Definition of a Limit

In this section we will discuss a better definition of a limit than the one given in earlier sections. If you remember in the earlier sections, the definitions were called "intuitive". We will have to analyze difficult functions and the definition given earlier just isn't good enough.

When we were discussing the limit of a function in previous sections, many of you might have asked yourself: "How close do we need to get to x so that the value of f(x) is close enough to L?" And then you may have thought, "What is 'close enough'?" The definition in this section will solve that problem. Let's look at the example below.

**Example:** $f(x) = \begin{Bmatrix} \frac{x^2-9}{x-3} & for\ x \neq 3 \\ 8 & for\ x = 3 \end{Bmatrix}$ This piece-wise function has one function for its entire domain except at ***x = 3***. When ***x = 3,*** the function value is 8. In section 2.3, we showed that when ***x*** is close to 3 but not equal to 3, then ***f(x)*** is close to 6. Therefore $\lim_{x \to 3} f(x) = 6$. But notice that when ***x = 3, f(x) = 8.*** So we might ask how close to 3 does x have to be so that f(x) is only 0.1 away from 6?

Mathematically we can write this using absolute values since x can be smaller or larger than 3.

- The distance from $x\ to\ 3 \Rightarrow |x-3|$
- The distance from $f(x) to\ 6\ is\ 0.1 \ \Rightarrow |f(x) - 6| < 0.1$

So if we can find a distance $\boldsymbol{\delta}$ (delta) between ***x*** and ***3*** such that $|f(x) - 6| < 0.1$ then we have our solution.

Mathematically speaking:

$$|f(x) - 6| < 0.1 \quad \boldsymbol{if} \quad 0 < |x-3| < \delta$$ **(It is important to note that | x – 3 | ≠ 0 since x ≠ 3)**

So our task is to find the value of $\boldsymbol{\delta}$ that makes $|f(x) - 6| < 0.1$. So we are going to find it by seeing if we can rewrite $|f(x) - 6| < 0.1$ to look like $|x-3| < \delta$.

$$|f(x) - 6| < 0.1 \qquad \Rightarrow |x-3| < \delta$$
$$\left|\frac{x^2-9}{x-3} - 6\right| < 0.1 \qquad \Rightarrow |x-3| < \delta$$
$$\left|\frac{x^2-9}{x-3} - \frac{6(x-3)}{x-3}\right| < 0.1 \Rightarrow |x-3| < \delta$$
$$\left|\frac{x^2-9-6x+18}{x-3}\right| < 0.1 \qquad \Rightarrow |x-3| < \delta$$
$$\left|\frac{x^2-6x+9}{x-3}\right| < 0.1 \qquad \Rightarrow |x-3| < \delta$$
$$\left|\frac{(x-3)\cancel{(x-3)}}{\cancel{x-3}}\right| < 0.1 \qquad \Rightarrow |x-3| < \delta$$
$$|x-3| < 0.1 \qquad \Rightarrow |x-3| < \delta \ \therefore\ \boldsymbol{\delta = 0.1}$$

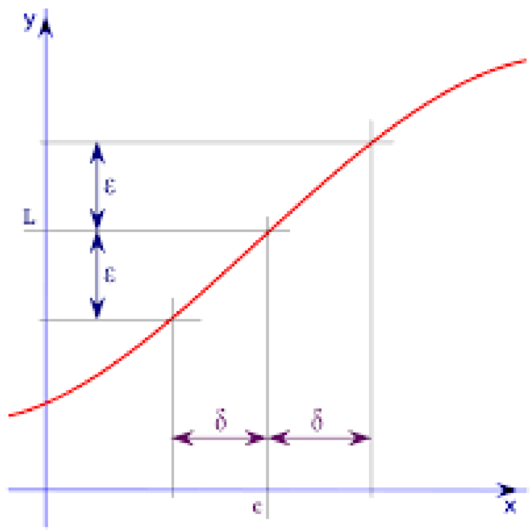
Therefore if ***x*** is within a distance of 0.1 from 3, then ***f(x)*** will be within a distance of 0.1 from 6.

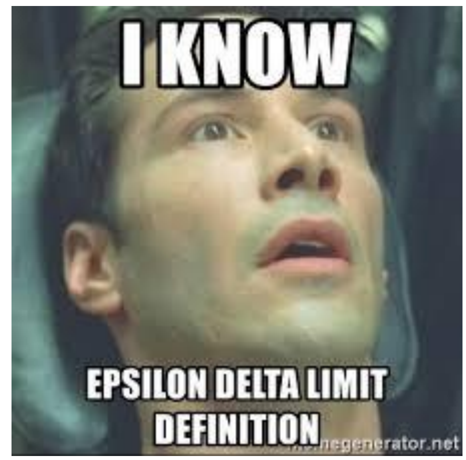
We could repeat this idea and make $|f(x) - 6| < 0.01$, we would just have to find a smaller $\delta$ value. Technically we could make $|f(x) - 6|$ be as small as we want by making $\delta$ smaller, but $\delta$ would depend on the distance of $|f(x) - 6|$.

Since we could make the $|f(x) - 6|$ arbitrarily small, let's denote that distance by $\boldsymbol{\varepsilon}$ (epsilon).
Thus:

$$|f(x) - 6| < \varepsilon \ \ if \ \ 0 < |x - 3| < \delta$$

Graphically it would look like: (where L = 6 and c = 3)




**The Precise Definition of a Limit:** Let ***f*** be a function defined on some open interval that contains the number ***a***, except possibly at ***a*** itself. Then we say that the limit of ***f(x)*** as $\boldsymbol{x}$ approaches ***a*** is ***L***, and we write it as:

| |
|---|
| $\lim_{x\to a} f(x) = L$ **if for every number** $\boldsymbol{\varepsilon > 0}$ **there is a number** $\boldsymbol{\delta > 0}$ **such that if** $0 < \lvert x - a\rvert < \delta$ **then** $\lvert f(x) - L\rvert < \varepsilon$ |

Notice that $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta \ \Rightarrow \ a + \delta < x < a + \delta$ and
$|f(x) - L| < \varepsilon$ is equivalent to $-\varepsilon < f(x) - L < \varepsilon \ \Rightarrow \ L - \varepsilon < f(x) < L + \varepsilon$

In terms of intervals, we have $\lim_{x\to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small $\varepsilon$ is) we can find a $\delta > 0$ such that if $\boldsymbol{x}$ lies in an open interval $(a - \delta, a + \delta)$ and $a \neq x$, then ***f(x)*** lies in an open interval $(L - \varepsilon, L + \varepsilon)$.

**Example:** Given: $f(x) = 4x + 5$. Prove that $\lim_{x\to 3}(4x - 5) = 7$

Let $\varepsilon > 0$. We want to find a number $\delta > 0$ such that if $|x - 3| < \delta$ then $|(4x - 5) - 7| < \varepsilon$

So… $|(4x - 5) - 7| = |4x - 5 - 7| = |4x - 12| = |4(x - 3)| = 4|x - 3|$.

Therefore, we want $\delta > 0$ such that if $0 < |x - 3| < \delta$ then $4|x - 3| < \varepsilon$ OR

If $0 < |x - 3| < \delta$ then $|x - 3| < \frac{\varepsilon}{4}$. Which means we can choose $\delta = \frac{\varepsilon}{4}$.

__Proof:__ Given $\varepsilon > 0$ and we choose $\delta = \frac{\varepsilon}{4}$. If $0 < |x - 3| < \delta$, then

$$\begin{aligned} |(4x - 5) - 7| &= |4x - 12| \\ &= 4|x - 3| < 4\delta \\ &= 4\left(\tfrac{\varepsilon}{4}\right) = \varepsilon \end{aligned}$$

Therefore, if $0 < |x-3| < \delta$ then $|(4x-5)-7| < \varepsilon$. And by definition of a limit, $\boldsymbol{\lim_{x\to 3}(4x-5) = 7.}$

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Just as we did before, let's define left – and right-hand limits.

**Precise Definition of Left-hand limit:**

$$\boldsymbol{\lim_{x\to a^-} f(x) = L}$$

If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $\boldsymbol{a-\delta < x < a}$ then $\boldsymbol{|f(x)-L| < \varepsilon.}$

**Precise Definition of Right-hand limit:**

$$\boldsymbol{\lim_{x\to a^+} f(x) = L}$$

If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $\boldsymbol{a < x < a+\delta}$ then $\boldsymbol{|f(x)-L| < \varepsilon.}$

When completing some these proofs, the triangle inequality might be necessary.

$$\boldsymbol{|a+b| \le |a| + |b|}$$

<u>**Infinite Limits**</u> Infinite limits can also be defined in a precise way.

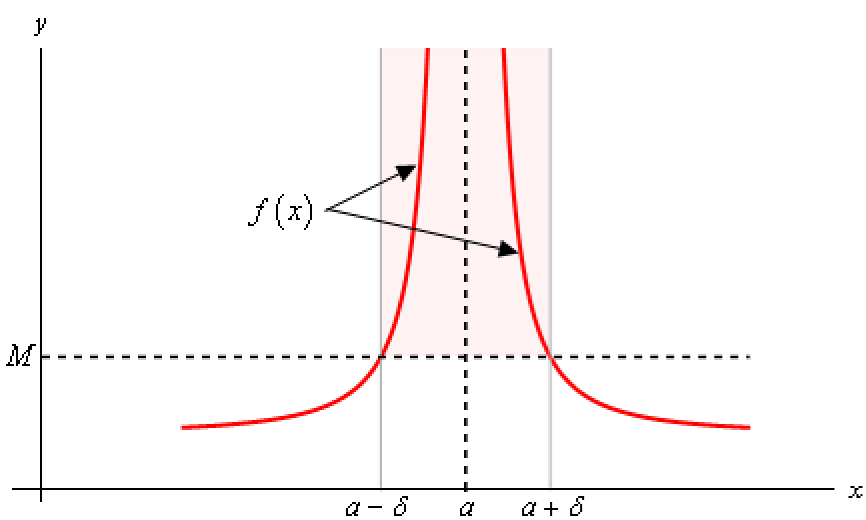
**Precise Definition of an Infinite Limit:** Let $\boldsymbol{f}$ be a function defined on some open interval that contains the number $\boldsymbol{a}$, except possibly at $\boldsymbol{a}$ itself. Then

$$\boldsymbol{\lim_{x\to a} f(x) = \infty}$$

Means that for every positive number **M** there is a positive number $\boldsymbol{\delta}$ such that if $\boldsymbol{0 < |x-a| < \delta}$ then $\boldsymbol{f(x) > M}$.

This means that for any $\boldsymbol{\delta}$ value we choose, there will always be a function value larger than **M** in the interval $(a-\delta, a+\delta)$, where **M** is the function value at $f(a-\delta)$ and $f(a+\delta)$.

Graphically this is what we have:




Similarly, we have the definition of a negative infinite limit.

**Precise Definition of an Infinite Limit:** Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

Means that for every negative number **N** there is a positive number $\delta$ such that if $0 < |x - a| < \delta$ then $f(x) < N$.

**Example:** Prove that $\lim_{x \to -1} \frac{1}{(x+1)^2} = \infty$ Let **M** be a positive large number. We want to find a $\delta$ such that

$$\text{If } |x - (-1)| < \delta \text{ then } \frac{1}{(x+1)^2} > M \quad \text{OR}$$

$$\text{If } |x + 1| < \delta \text{ then } \frac{1}{(x+1)^2} > M$$

Notice that $\frac{1}{M} > (x+1)^2 \Leftrightarrow \sqrt{\frac{1}{M}} > \sqrt{(x+1)^2} \Leftrightarrow \sqrt{\frac{1}{M}} > |x+1|$

so we could choose $\delta = \sqrt{\frac{1}{M}}$ and $|x+1| < \sqrt{\frac{1}{M}}$ since $|x+1| < \delta$.

Using algebra we can rewrite $(x+1)^2 < \frac{1}{M}$ as $M < \frac{1}{(x+1)^2}$. This shows that $\frac{1}{(x+1)^2} \to \infty$ as $x \to -1$.