12.2) Introduction to Vectors

1. Oriented Line Segments:

In geometry, a **line segment** is defined as a subset of a line consisting of two distinct points of the line, known as the segment's **endpoints**, and all points of the line lying inbetween those two endpoints. We shall refer to this as the *classical* definition of a line segment.

For our purposes, it is useful to adopt a slightly more general definition. We shall omit the stipulation that the two endpoints must be *distinct* points–i.e., we shall allow for the possibility that they could be the *same* point. In this case, there are no points of the line lying inbetween the two endpoints, and the line segment consists of a single point of the line. (In contrast, when the two endpoints are distinct, the line segment consists of infinitely many points, because there are infinitely many points between the two endpoints.)

We shall refer to a line segment with two distinct endpoints as a **proper line segment**, and we shall refer to a line segment with only one distinct endpoint as an **improper line segment**. (Caution: "improper" does not mean illegitimate. An improper line segment is a legitimate line segment, according to our definition, just as, in arithmetic, an improper fraction is a legitimate fraction, and, in set theory, an improper subset is a legitimate subset.)

The **length** of a line segment is the *distance* between its two endpoints. The length of a proper line segment is a positive real number, and the length of an improper line segment is zero.

An **oriented line segment** is a line segment where one endpoint has been designated the **initial point** and the other endpoint has been designated the **terminal point**. If the line segment is proper, the initial and terminal points are distinct points, but if the line segment is improper, then they are the same point.

We say an oriented line segment extends *from* its initial point *to* its terminal point.

The initial point is also known as the **tail**. The terminal point is also known as the the **tip** or **head**.

If an oriented line segment is proper, then, when we draw the line segment, we place an *arrowhead* at the terminal point. We do not use an arrowhead when drawing an improper oriented line segment; such a segment consists of a single point, which is depicted as a *dot*.

We name points with capital letters. If an oriented line segment has initial point A and terminal point B, then it may be denoted as $\overrightarrow{AB}$. (In this notation, we always write the initial point first and the terminal point second.) This is a symbolic representation of the oriented line segment, not to be confused with an actual drawing of the segment itself.

Bear in mind, we are allowing for the possibility that the initial and terminal points of an oriented line segment may be the *same* point. If we name the former point A and the latter point $B,$ we are *not* implying the points are *distinct*. We may have $A = B,$ in which case $\overrightarrow{AB}$ is an improper oriented line segment, or we may have $A \neq B,$ in which case $\overrightarrow{AB}$ is a proper oriented line segment. (If $A = B,$ then $\overrightarrow{AB}$ could also be written as $\overrightarrow{AA}$ or $\overrightarrow{BB}.$)

The length of $\overrightarrow{AB}$ is denoted $\left|\overrightarrow{AB}\right|$. This is also referred to as the **magnitude** of $\overrightarrow{AB}$. (Some books use the notation $\left\|\overrightarrow{AB}\right\|$.)

If an oriented line segment is proper, then it points in a particular direction, which we refer to as the **direction** of the segment. A proper oriented line segment may thus also be referred to as a **directed line segment**. In contrast, an improper oriented line segment has *no direction* (we can also say its direction is *undefined*). In the notation $\overrightarrow{AB},$ the arrow symbol always points rightward, but this does not necessarily reflect the actual direction of the segment itself (or even if it has a direction).

(Caution: In geometry, there is something called a **ray**. This is a subset of line consisting of one point of the line, known as the *vertex*, and all points of the line lying on one particular side of the vertex. Whereas a line extends infinitely far in two opposite directions, a ray extends infinitely far in only one direction. In a geometry class, the notation $\overrightarrow{AB}$ would represent a ray, rather than an oriented line segment. In this class, the notation will always refer to an oriented line segment, never a ray. In fact, we will never deal with rays at all in this class. If you see a drawing of a line segment with an arrowhead at one end, it should always be interpreted as a directed line segment, rather than a ray.)

Oriented line segments can exist in one-dimensional space, two-dimensional space, or three-dimensional space. (Higher-dimensional spaces can be theorized, but they cannot be visualized, because we have only three physical dimensions available to us.)

One-dimensional space is simply the traditional number line. In this setting, a directed line segment can have only two possible directions. Assuming the number line is drawn horizontally, the two possible directions are *leftward* and *rightward*. If the terminal point lies to the right of the initial point, then the direction is rightward, whereas if the terminal point lies to the left of the initial point, then the direction is leftward.

In two-dimensional space or three-dimensional space, a directed line segment can have *infinitely many* possible directions. A precise direction can be specified through the use of angles. In Section 12.3, we will develop this concept rigorously; for now, we assume the concept of direction is intuitively clear.

In two-dimensional or three-dimensional space, we shall adopt the **Cartesian coordinate system**. In two-dimensional space, we have two *coordinate axes*, the x axis (which is horizontal) and the y axis (which is vertical); this space is thus referred to as the x,y plane. In three-dimensional space, we have three *coordinate axes*, the x and y axes (which are perpendicular to each other and which lie in a horizontal plane) and the z axis (which is vertical); this space is thus referred to as x,y,z space.

- Every point in the x,y plane corresponds to a unique *ordered pair* of real numbers, which are known as the point's x and y *coordinates*; for instance, the ordered pair (x_1,y_1) represents the point in the x,y plane that aligns perpendicularly with the value x_1 on the x axis and with the value y_1 on the y axis. The two axes intersect at the point representing 0 on each axis; this point, represented by the ordered pair $(0,0)$, is called the *origin* and is usually named by the letter O.
- Every point in x,y,z space corresponds to a unique *ordered triple* of real numbers, which are known as the point's x, y, and z *coordinates*; for instance, the ordered triple (x_1,y_1,z_1) represents the point in x,y,z space that aligns perpendicularly with the value x_1 on the x axis, with the value y_1 on the y axis, and with the value z_1 on the z axis. The three axes intersect at the point representing 0 on each axis; this point, represented by the ordered triple $(0,0,0)$, is called the *origin* and is usually named by the letter O.

An oriented line segment is said to be a **standard-position line segment** if its initial point is the origin. An oriented line segment whose initial point is not the origin is said to be a **non-standard-position line segment**.

Example One: In the x,y plane, the oriented line segment from $(0,0)$ to $(-7,-4)$ is a standard-position line segment, whereas the oriented line segments from $(3,-1)$ to $(5,4)$ and from $(6,2)$ to $(0,0)$ are non-standard-position line segments. Furthermore, the oriented line segment from $(0,0)$ to $(0,0)$ is a standard-position line segment, whereas the oriented line segment from $(3,-1)$ to $(3,-1)$ is a non-standard-position line segment. In other words, if $O = (0,0)$, $A = (-7,-4)$, $B = (3,-1)$, $C = (5,4)$, and $D = (6,2)$, then $\overrightarrow{OA}$ and $\overrightarrow{OO}$ are standard-position line segments, whereas $\overrightarrow{BC}$, $\overrightarrow{DO}$ and $\overrightarrow{BB}$ are non-standard-position line segments.

2. Algebraic Formulas for Oriented Line Segments:

In the x,y plane, if point A has coordinates (x_1,y_1) and point B has coordinates (x_2,y_2), then, by the Distance Formula, $\left|\overrightarrow{AB}\right| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. Notice that if $A = B$, then $x_1 = x_2$ and $y_1 = y_2$, in which case the formula gives us a result of 0, but if $A \neq B$, then $x_1 \neq x_2$ or $y_1 \neq y_2$, and the result is a positive real number.

In x,y,z space, if point A has coordinates (x_1,y_1,z_1) and point B has coordinates (x_2,y_2,z_2), then, by the Distance Formula, $\left|\overrightarrow{AB}\right| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$. Notice that if $A = B$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$, in which case the formula gives us a result of 0, but if $A \neq B$, then $x_1 \neq x_2$ or $y_1 \neq y_2$ or $z_1 \neq z_2$, and the result is a positive real number.

Example Two: In the x,y plane, if $A = (2,-7)$ and $B = (-3,-1)$, then $\left|\overrightarrow{AB}\right| = \sqrt{(-3-2)^2+(-1--7)^2} = \sqrt{(-5)^2+6^2} = \sqrt{61}$.

Example Three: In x,y,z space, if $A = (1,4,7)$ and $B = (3,7,13)$, then $\left|\overrightarrow{AB}\right| = \sqrt{(3-1)^2 + (7-4)^2 + (13-7)^2} = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7.$

In the Cartesian coordinate system, geometrical objects can be represented by algebraic equations. For instance, in the x,y plane, the equation $2x - 3y = 6$ represents a line, the equation $y = x^2$ represents a parabola, and the equation $x^2 + y^2 = 25$ represents a circle. Likewise, directed line segments can be represented by algebraic equations. Before we address this topic, we must first discuss how to write **parametric equations** for a line in either two or three dimensions.

In the x,y plane, let (x_1,y_1) and (x_2,y_2) be two distinct points on a line. Let $a = x_2 - x_1$, and let $b = y_2 - y_1$. Because (x_1,y_1) and (x_2,y_2) are distinct, a and b cannot *both* be zero. (If a is nonzero, then the line is nonvertical and has slope $\frac{b}{a}$.) The line can be represented *parametrically* by the equations $x = x_1 + at,\ y = y_1 + bt,$ where $t \in (-\infty,\infty)$. The variable t is known as an *independent parameter*.

In x,y,z space, let (x_1,y_1,z_1) and (x_2,y_2,z_2) be two distinct points on a line. Let $a = x_2 - x_1$, let $b = y_2 - y_1$, and let $c = z_2 - z_1$. Because (x_1,y_1,z_1) and (x_2,y_2,z_2) are distinct, a, b, and c cannot *all* be zero. The line can be represented *parametrically* by the equations $x = x_1 + at$, $y = y_1 + bt,\ z = z_1 + ct,$ where $t \in (-\infty,\infty)$. The variable t is known as an *independent parameter*.

Once a line in either two or three dimensions has been parameterized in accordance with the above equations, every point on the line corresponds to a unique real value of t, and every real value of t corresponds to a unique point on the line.

Since a directed line segment is a subset of a line, the parametric equations for the line can likewise serve as parametric equations for the segment; we need only restrict the values of the parameter t.

In the x,y plane, let $A = (x_1,y_1)$ and $B = (x_2,y_2)$ be two distinct points. Let $a = x_2 - x_1$, and let $b = y_2 - y_1$. Then the directed line segment $\overrightarrow{AB}$ has parametric equations $x = x_1 + at$, $y = y_1 + bt,$ where $t \in [0,1]$. Notice that the initial point A is generated when $t = 0$ and the terminal point B is generated when $t = 1$. Values of t between 0 and 1 generate points on $\overrightarrow{AB}$ between A and B.

In x,y,z space, let $A = (x_1,y_1,z_1)$ and $B = (x_2,y_2,z_2)$ be two distinct points. Let $a = x_2 - x_1$, let $b = y_2 - y_1$, and let $c = z_2 - z_1$. Then the directed line segment $\overrightarrow{AB}$ has parametric equations $x = x_1 + at,\ y = y_1 + bt,\ z = z_1 + ct,$ where $t \in [0,1]$. Notice that the initial point A is generated when $t = 0$ and the terminal point B is generated when $t = 1$. Values of t between 0 and 1 generate points on $\overrightarrow{AB}$ between A and B.

Example Four: In the x,y plane, let $A = (-3,5)$ and $B = (6,-1)$. $a = 6 - -3 = 9$ and $b = -1 - 5 = -6$. So the directed line segment $\overrightarrow{AB}$ has parametric equations $x = -3 + 9t$, $y = 5 - 6t,$ where $t \in [0,1]$.

Example Five: In x,y,z space, let $A = (4,-1,-9)$ and $B = (-2,-6,3)$. $a = -2-4 = -6$, $b = -6--1 = -5$, and $c = 3--9 = 12$. So the directed line segment $\overrightarrow{AB}$ has parametric equations $x = 4-6t,\ y = -1-5t,\ z = -9+12t$, where $t \in [0,1]$.

3. Equivalence of Oriented Line Segments:

Two oriented line segments in the same dimensional setting are said to be **equivalent** to each other if they have the same length and if they have the same direction (or both have no direction).

Example Six: Given a horizontal number line, the following four directed line segments are equivalent, because they all have length 6 and are all directed leftward:

- The directed line segment whose initial point is at 15 and whose terminal point is at 9.
- The directed line segment whose initial point is at 2 and whose terminal point is at -4.
- The directed line segment whose initial point is at 0 and whose terminal point is at -6.
- The directed line segment whose initial point is at -7 and whose terminal point is at -13.

Example Seven: In the x,y plane, let $A = (0,0),\ B = (3,4),\ C = (6,0)$, and $D = (9,4)$. Then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent, because they both have length 5, and they both have the same direction. (We have not yet rigorously discussed direction, but notice that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel, because they both have slope $\frac{4}{3}$, and so these lines have the same angles relative to the x and y axes.)

Can we find a general principle to determine if two oriented line segments are equivalent? Yes we can:

- In the x,y plane, let $A = (x_1,y_1),\ B = (x_2,y_2),\ C = (x_3,y_3)$, and $D = (x_4,y_4)$. Then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if and only if $x_4 - x_3 = x_2 - x_1$ and $y_4 - y_3 = y_2 - y_1$.
- In x,y,z space, let $A = (x_1,y_1,z_1),\ B = (x_2,y_2,z_2),\ C = (x_3,y_3,z_3)$, and $D = (x_4,y_4,z_4)$. Then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if and only if $x_4 - x_3 = x_2 - x_1$ and $y_4 - y_3 = y_2 - y_1$ and $z_4 - z_3 = z_2 - z_1$.

Example Eight: In x,y,z space, if $A = (3,-7,5),\ B = (-2,8,10),\ C = (-5,-4,-1)$, and $D = (-10,11,4)$, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent, because $-10--5 = -2-3$, $11--4 = 8--7$, and $4--1 = 10-5$.

4. Vectors Defined As Equivalence Classes:

A **vector** is an *equivalence class* of oriented line segments. In other words, it is a set consisting of all oriented line segments having a particular length and a particular direction (or no direction); all segments in the set are equivalent to each other, and every segment equivalent to any segment in the set is included in the set. Each vector encompasses infinitely many oriented line segments, all of which have different initial points and different terminal points. These oriented line segments may be referred to as *members* of the equivalence class or as *representations* of the vector.

This may be the first time you have been formally introduced to the concept of an equivalence class. However, it is a concept you are *informally* familiar with, from basic arithmetic. A rational number is an equivalence class of fractions. In other words, it is a set consisting of all fractions having a particular ratio between the numerator and the denominator; all fractions in the set are equivalent to each other, and every fraction equivalent to any fraction in the set is included in the set. Each rational number encompasses infinitely many fractions, all of which have different numerators and denominators. These fractions may be referred to as members of the equivalence class or as representations of the rational number. For instance, the rational number “one half” is the equivalence class $\{\frac{1}{2}, \frac{-1}{-2}, \frac{2}{4}, \frac{-2}{-4}, \frac{3}{6}, \frac{-3}{-6}, \frac{4}{8}, \frac{-4}{-8}, \ldots\}$.

When we are dealing with two equivalent oriented line segments, we can say, speaking formally, that they *represent* the same vector, or *are representations of* the same vector; however, speaking informally, we can say that they *are the same vector*. (But you must bear in mind that the vector is not just these two oriented line segments; it comprises infinitely many oriented line segments.)

Recall that oriented line segments can exist in one-dimensional space, two-dimensional space, or three-dimensional space. Consequently, vectors can be classified as *one-dimensional vectors*, *two-dimensional vectors*, or *three-dimensional vectors*. Henceforth, we will pay no further attention to one-dimensional vectors; we will study only two-dimensional and three-dimensional vectors.

As noted above, a vector is an equivalence class comprising infinitely many oriented line segments, all of which have different initial points. Exactly one of these has its initial point at the origin (i.e., exactly one is a standard-position line segment, and all other members of the class are non-standard-position line segments). We refer to this unique member of the class as the vector’s **standard-position representation**. It is also referred to as the vector **in standard position**.

If we know any representation for a vector, we can find the vector’s standard-position representation. We already know that the initial point will be the origin, so we only need calculate its terminal point. Let us refer to the terminal point as (a,b) in the two-dimensional case and (a,b,c) in the three-dimensional case. To compute these coordinates, we can apply the formulas discussed earlier.

- In the x,y plane, if $A = (x_1,y_1)$, $B = (x_2,y_2)$, $C = (x_3,y_3)$, and $D = (x_4,y_4)$, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if and only if $x_4 - x_3 = x_2 - x_1$ and $y_4 - y_3 = y_2 - y_1$. Suppose $\overrightarrow{AB}$ is known and $\overrightarrow{CD}$ is the standard-position representation that we wish to find. We substitute (a,b) for (x_4,y_4) and $(0,0)$ for (x_3,y_3), giving us the following equations:
  1. $a = x_2 - x_1$
  2. $b = y_2 - y_1$

- In x,y,z space, if $A = (x_1,y_1,z_1)$, $B = (x_2,y_2,z_2)$, $C = (x_3,y_3,z_3)$, and $D = (x_4,y_4,z_4)$, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if and only if $x_4 - x_3 = x_2 - x_1$ and $y_4 - y_3 = y_2 - y_1$ and $z_4 - z_3 = z_2 - z_1$. Suppose $\overrightarrow{AB}$ is known and $\overrightarrow{CD}$ is the standard-position representation that we wish to find. We substitute (a,b,c) for (x_4,y_4,z_4) and $(0,0,0)$ for (x_3,y_3,z_3), giving us the following equations:
  1. $a = x_2 - x_1$
  2. $b = y_2 - y_1$
  3. $c = z_2 - z_1$

Example Nine: If one representation of a three-dimensional vector has initial point $(4,-5,11)$ and terminal point $(-2,6,7)$, then the vector in standard position will have terminal point $(a,b,c) = (-2-4, 6--5, 7-11) = (-6,11,-4)$.

5. Component Notation for Vectors:

If a two-dimensional vector in standard position has terminal point (a,b), then the vector may be denoted $< a,b >$. This is known as **component notation** or **component form** for the vector. Bear in mind, this notation refers to the vector itself (i.e., the entire equivalence class), not just to the vector's standard-position representation. Thus, the notation $< a,b >$ may be applied to each of the infinitely many oriented line segments belonging to the equivalence class.

Similarly, if a three-dimensional vector in standard position has terminal point (a,b,c), then the vector may be denoted $< a,b,c >$.

In the notation $< a,b >$, we refer to a as the *first component* and to b as the *second component*.

In the notation $< a,b,c >$, we refer to a as the *first component,* to b as the *second component*, and to c as the *third component*.

The magnitude of the vector $< a,b >$ is $\sqrt{a^2+b^2}$. This may be denoted $|< a,b >|$.

The magnitude of the vector $< a,b,c >$ is $\sqrt{a^2+b^2+c^2}$. This may be denoted $|< a,b,c >|$.

Example 10: The magnitude of the vector $< 2,-6,4 >$ is $\sqrt{2^2+(-6)^2+4^2} = \sqrt{56} = 2\sqrt{14}$.

If a two-dimensional vector has a representation where the initial point is (x_1,y_1) and the terminal point is (x_2,y_2), then its component form, $< a,b >$, can be found from the equations $a = x_2 - x_1$ and $b = y_2 - y_1$.

If a three-dimensional vector has a representation where the initial point is (x_1,y_1,z_1) and the terminal point is (x_2,y_2,z_2), then its component form, $< a,b,c >$, can be found from the equations $a = x_2 - x_1$ and $b = y_2 - y_1$ and $c = z_2 - z_1$.

Example 11: In the x,y plane, if $A = (4,6)$ and $B = (7,1)$, then $\overrightarrow{AB}$ has component form $< 7-4, 1-6 > = < 3,-5 >$. In x,y,z space, if $A = (3,-7,5)$ and $B = (-2,8,10)$, then $\overrightarrow{AB}$ has component form $< -2-3,\ 8--7,\ 10-5 > = < -5,15,5 >$.

For any vector $< a,b >$ and any point (x_0,y_0) in the x,y plane, the vector has exactly one representation whose initial point is (x_0,y_0). The terminal point of this representation is the point (x_0+a, y_0+b). We refer to this representation as the vector **placed at** (x_0,y_0).

For any vector $< a,b,c >$ and any point (x_0,y_0,z_0) in x,y,z space, the vector has exactly one representation whose initial point is (x_0,y_0,z_0). The terminal point of this representation is the point (x_0+a, y_0+b, z_0+c). We refer to this representation as the vector **placed at** (x_0,y_0,z_0).

Example 12: If the vector $< -8,14 >$ is placed at $(3,-9)$, then its terminal point will be $(3-8,-9+14) = (-5,5)$. If the vector $< 2,-1,7 >$ is placed at the point $(-9,13,-11)$, then its terminal point will be $(-9+2, 13+-1, -11+7) = (-7,12,-4)$.

Given any point (x_1,y_1) in the x,y plane, we may refer to the vector $< x_1,y_1 >$ as the **position vector** for that point. Given any point (x_1,y_1,z_1) in x,y,z space, we may refer to the vector $< x_1,y_1,z_1 >$ as the **position vector** for that point. The position vector for any point is usually depicted in standard position, so the initial point is the origin and the terminal point is the given point. (Of course, the vector has infinitely many representations, but if we are thinking of the vector *as the position vector* for the specified point, then the vector's standard-position representation is the one we would want to depict.)

Example 13: The position vector for the point $(-2,5)$ in the x,y plane is $< -2,5 >$, which we would depict as the directed line segment $\overrightarrow{OA}$, where $O = (0,0)$ and $A = (-2,5)$.

6. Vector and Scalar Symbols:

A real number value is known as a **scalar**.

A vector or a scalar may be represented by a single letter; in the case of a scalar, we almost always use a lowercase letter, but in the case of a vector, we may use either a lowercase or a capital letter (with the former being more common). In printed text, a letter representing a vector appears in *boldface type*, while a letter representing a scalar appears in *normal type*. When writing by hand, we cannot use boldface, so we indicate that a letter represents a vector by affixing the "hat" symbol atop the letter. For example, a vector may be expressed as $\mathbf{w}$ in printed text or as $\widehat{w}$ by hand.

Once we have named a vector with a particular letter (in boldface type or with the hat), then we may use that same letter in normal type and without the hat to represent the *magnitude* of that vector. In other words, $w = |\mathbf{w}| = |\widehat{w}|$. For example, if $\mathbf{w} = < 5,-12 >$, then $w = 13$. Furthermore, we may affix *subscripts* to the given letter (in normal type and without the hat) to represent the vector's components. In the preceding example, we would have $w_1 = 5$ and $w_2 = -12$. More generally, if $\mathbf{a}$ is a two-dimensional vector, then $\mathbf{a} = < a_1, a_2 >$, and if $\mathbf{a}$ is a three-dimensional vector, then $\mathbf{a} = < a_1, a_2, a_3 >$.

When reading aloud, $\mathbf{a}$ may be pronounced "vector a" and a may be pronounced "scalar a."

For a two-dimensional vector $\mathbf{a}$, $a = \sqrt{(a_1)^2 + (a_2)^2}$. For a three-dimensional vector $\mathbf{a}$, $a = \sqrt{(a_1)^2 + (a)^2 + (a_3)^2}$.

Example 14: If $\mathbf{a} = < 9, -6 >$, then $a = \sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$. If $\mathbf{a} = < 2, 7, -1 >$, then $a = \sqrt{2^2 + 7^2 + (-1)^2} = \sqrt{54} = 3\sqrt{6}$.

In either two dimensions or in three dimensions, the **zero vector** is the equivalence class consisting of all oriented line segments having zero length and no direction–i.e., all oriented line segments where the initial point is also the terminal point (in other words, all improper oriented line segments). The zero vector is denoted $\mathbf{0}$ or $\widehat{0}$. Every representation of the zero vector (i.e., every member of the equivalence class) consists of a single point and is depicted as a dot. In standard position, this point is the origin. In component form, the two-dimensional zero vector is $< 0, 0 >$ and the three-dimensional zero vector is $< 0, 0, 0 >$.

The zero vector is the *only* vector whose magnitude is 0, and it is the *only* vector having no direction. Every other vector is referred to as a **nonzero vector**. Every nonzero vector has a positive magnitude and has a direction, and is depicted as a directed line segment.

Two vectors are **equal** to each other if and only if their corresponding components are equal.

- For two-dimensional vectors $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$, $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$ and $a_2 = b_2$.
- For three-dimensional vectors $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$, $\mathbf{a} = \mathbf{b}$ if and only if $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$.

7. Operations on Vectors:

We now consider various operations that can be performed involving vectors. Further operations will be studied in Sections 12.3 and 12.4.

Scalar multiplication of a vector (i.e., **multiplying a vector by a scalar**):

- If $\mathbf{a} = < a_1, a_2 >$, then $c\mathbf{a} = c < a_1, a_2 > = < ca_1, ca_2 >$.
- If $\mathbf{a} = < a_1, a_2, a_3 >$, then $c\mathbf{a} = c < a_1, a_2, a_3 > = < ca_1, ca_2, ca_3 >$.

Note that the result is a vector. The product $c\mathbf{a}$ is referred to as a **scalar multiple** of $\mathbf{a}$.

Whenever we have are multiplying a scalar and a vector, we generally write the scalar before the vector. In other words, we write $c\mathbf{a}$ rather than $\mathbf{a}c$. There are exceptions, however. (For instance, if $\mathbf{v}$ is a velocity vector and dt is the differential of time, then we would write $\mathbf{v}\,dt$, rather than the other way around.)

Example 15: $4 < 3,-7 > = < 12,-28 >$, and $-3 < 5,-2,8 > = < -15,6,-24 >$.

Vector addition and subtraction:

- If $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$, then
  $\mathbf{a} + \mathbf{b} = < a_1 + b_1, a_2 + b_2 >$, and
  $\mathbf{a} - \mathbf{b} = < a_1 - b_1, a_2 - b_2 >$.
- If $\mathbf{a} = < a_1, a_2, a_3 >$ and $\mathbf{b} = < b_1, b_2, b_3 >$, then
  $\mathbf{a} + \mathbf{b} = < a_1 + b_1, a_2 + b_2, a_3 + b_3 >$, and
  $\mathbf{a} - \mathbf{b} = < a_1 - b_1, a_2 - b_2, a_3 - b_3 >$.

Note that the result of vector addition or subtraction is a vector.

Example 16: $< 5,9 > + < 3,-2 > = < 8,7 >$, and $< -4,3,11 > - < -6,7,6 > = < 2,-4,5 >$.

Note that $\mathbf{a} + \mathbf{a} = 2\mathbf{a}$, $\mathbf{a} + \mathbf{a} + \mathbf{a} = 3\mathbf{a}$, and so on.

These operations have the following properties:

- $0\mathbf{a} = \mathbf{0}$. (The scalar zero times any vector gives us the zero vector.)
- $c\mathbf{0} = \mathbf{0}$. (Any scalar times the zero vector gives us the zero vector.)
- If $c\mathbf{a} = \mathbf{0}$, then either $c = 0$ or $\mathbf{a} = \mathbf{0}$. (If a scalar multiple of a vector gives us the zero vector, then either that scalar or that vector must be zero.)
- Equivalently: If $c \neq 0$ and $\mathbf{a} \neq \mathbf{0}$, then $c\mathbf{a} \neq \mathbf{0}$. (The product of a nonzero scalar and a nonzero vector must be a nonzero vector.)
- $1\mathbf{a} = \mathbf{a}$. (The scalar 1 times any vector gives us that same vector.)
- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, but $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$ (Vector addition is commutative, but vector subtraction is not.)
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$, but $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$ (Vector addition is associative, but vector subtraction is not.)
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$. (Any vector plus the zero vector is equal to itself.)
- $\mathbf{a} - \mathbf{a} = \mathbf{0}$. (Any vector subtracted from itself is equal to the zero vector.)
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$, and $c(\mathbf{a} - \mathbf{b}) = c\mathbf{a} - c\mathbf{b}$ (Scalar multiplication of a vector distributes over vector addition and subtraction.)
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$, and $(c - d)\mathbf{a} = c\mathbf{a} - d\mathbf{a}$ (Scalar multiplication of a vector distributes over scalar addition and subtraction.)
- $(cd)\mathbf{a} = c(d\mathbf{a})$ (Scalar multiplication of a vector is associative.)

For any given vector, there is a unique vector that can be added to it so that the sum of the two vectors is $\mathbf{0}$. We call this the **additive inverse** of the given vector. The additive inverse of a vector $\mathbf{a}$ is denoted $-\mathbf{a}$.

- The additive inverse of $< a_1, a_2 >$ is $< -a_1, -a_2 >$, because $< a_1, a_2 > + < -a_1, -a_2 > =$ $< a_1 - a_1, a_2 - a_2 > = \ < 0,0 >$.
- The additive inverse of $< a_1, a_2, a_3 >$ is $< -a_1, -a_2, -a_3 >$, because $< a_1, a_2, a_3 > + < -a_1, -a_2, -a_3 > = < a_1 - a_1, a_2 - a_2, a_3 - a_3 > = \ < 0,0,0 >$.

Example 17: The additive inverse of $< 2,-5,4 >$ is $- < 2,-5,4 > = < -2,5,-4 >$.

The additive inverse of a given vector is also referred to as the **negative** of that vector, or as the **opposite** of that vector, or as the **opposite vector**.

Properties:

- $-(-\mathbf{a}) = \mathbf{a}$
- $-\mathbf{0} = \mathbf{0}$
- $-\mathbf{a} = -1\mathbf{a}$
- $\mathbf{a} - \mathbf{b} = \mathbf{a} + -\mathbf{b}$
- $\mathbf{b} - \mathbf{a} = -(\mathbf{a} - \mathbf{b}) = -1(\mathbf{a} - \mathbf{b})$

The zero vector is the only vector that is its own opposite. Any nonzero vector and its opposite vector are two distinct vectors.

Any nonzero vector and its opposite vector have *opposite directions*. (This idea will be elaborated upon in Section 12.3, where we rigorously address the subject of direction.) The concept of "opposite direction" cannot be applied to the zero vector, since it has no direction.

For any nonzero vector $\mathbf{a}$ and any nonzero scalar c, $c\mathbf{a}$ has the *same direction* as $\mathbf{a}$ when c is positive, whereas $c\mathbf{a}$ has the *opposite direction* from $\mathbf{a}$ when c is negative. If $|c| < 1$, then $c\mathbf{a}$ is shorter than $\mathbf{a}$. If $|c| > 1$, then $c\mathbf{a}$ is longer than $\mathbf{a}$. If $|c| = 1$, then $c\mathbf{a}$ has the same length as $\mathbf{a}$.

For any vector $\mathbf{a}$ and any scalar c, $|c\mathbf{a}| = |c||\mathbf{a}| = |c|a$. ($|c|$ denotes the absolute value of c.)

Example 18: If $\mathbf{a} = < -6,8 >$, then $\frac{1}{2}a = < -3,4 >$ and $-3\mathbf{a} = < 18,-24 >$. You may confirm that the lengths of these three vectors are, respectively, 10, 5, and 30.

Vector addition and subtraction (for nonzero vectors) have interesting geometric interpretations, which we will discuss in class. Here's a summary of what we will discuss:

- Vector addition by the **tip to tail method** (better thought of as **tail on tip**, because we place the *tail* of the second vector *on the tip* of the first). This is also known as the **triangle method of addition**.
- Vector subtraction done the same way, but replacing the second vector with its opposite vector.
- Vector subtraction by the **tip to tip** (or **tail on tail**) **method**, also known as the **triangle method of subtraction**.
- Vector addition and subtraction by the **parallelogram method**.

For any scalars c and d, $c\mathbf{a} + d\mathbf{b}$ is known as a **linear combination** of the vectors $\mathbf{a}$ and $\mathbf{b}$.

- If $\mathbf{a} = < a_1,a_2 >$ and $\mathbf{b} = < b_1,b_2 >$, then $c\mathbf{a} + d\mathbf{b} = < ca_1 + db_1, ca_2 + db_2 >$.
- If $\mathbf{a} = < a_1,a_2,a_3 >$ and $\mathbf{b} = < b_1,b_2,b_3 >$, then $c\mathbf{a} + d\mathbf{b} =$ $< ca_1 + db_1, ca_2 + db_2, ca_3 + db_3 >$.

Note that the result is a vector.

Example 19: If $\mathbf{a} = < 7,6 >$ and $\mathbf{b} = < 2,5 >$, then $4\mathbf{a} + 9\mathbf{b} = < 28,24 > + < 18,45 > =$ $< 46,69 >$. Thus, $< 46,69 >$ is a linear combination of $< 7,6 >$ and $< 2,5 >$.

8. Unit Vectors:

Any vector whose magnitude is 1 is called a **unit vector**.

Example 20: $< \frac{3}{5}, \frac{4}{5} >$ and $< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$ are unit vectors, as you may verify by computing their lengths.

For any nonzero vector $\mathbf{a}$, there is a unique unit vector having the *same direction* as $\mathbf{a}$, and there is a unique unit vector having the *opposite direction* from $\mathbf{a}$. The former unit vector is $\frac{1}{a}\mathbf{a}$, and the latter is $-\frac{1}{a}\mathbf{a}$. These vectors can also be written as $\frac{\mathbf{a}}{a}$ and $-\frac{\mathbf{a}}{a}$.

- If $\mathbf{a} = < a_1, a_2 >$ is nonzero, then $\frac{\mathbf{a}}{a} = < \frac{a_1}{a}, \frac{a_2}{a} >$ and $-\frac{\mathbf{a}}{a} = < \frac{-a_1}{a}, \frac{-a_2}{a} >$.
- If $\mathbf{a} = < a_1, a_2, a_3 >$ is nonzero, then $\frac{\mathbf{a}}{a} = < \frac{a_1}{a}, \frac{a_2}{a}, \frac{a_3}{a} >$ and $-\frac{\mathbf{a}}{a} = < \frac{-a_1}{a}, \frac{-a_2}{a}, \frac{-a_3}{a} >$.

Example 21: For the vector $< 3,-8 >$, the unit vector in the same direction is $< \frac{3}{\sqrt{73}}, \frac{-8}{\sqrt{73}} >$, and the unit vector in the opposite direction is $< \frac{-3}{\sqrt{73}}, \frac{8}{\sqrt{73}} >$. For the vector $< -6,2,-3 >$, the unit vector in the same direction is $< \frac{-6}{7}, \frac{2}{7}, \frac{-3}{7} >$, and the unit vector in the opposite direction is $< \frac{6}{7}, \frac{-2}{7}, \frac{3}{7} >$.

For any nonzero vector $\mathbf{a}$ and any positive scalar c, the vector $\frac{c}{a}\mathbf{a}$ has length c and has the same direction as $\mathbf{a}$, while the vector $-\frac{c}{a}\mathbf{a}$ has length c and has the opposite direction from $\mathbf{a}$. These vectors can also be written as $c\frac{\mathbf{a}}{a}$ and $-c\frac{\mathbf{a}}{a}$. (Each vector has length c because $|\pm c\frac{\mathbf{a}}{a}| = |\pm c||\frac{\mathbf{a}}{a}| = c(1) = c$.)

Example 22: Let $\mathbf{a} = < -4,3 >$, so $a = 5$. $\frac{7}{5}\mathbf{a}$ has length 7 and has the same direction as $\mathbf{a}$, while $-\frac{7}{5}\mathbf{a}$ has length 7 and has the opposite direction from $\mathbf{a}$.

In two-dimensional space, there are *two* unit vectors that are special. They are $< 1,0 >$, which is denoted $\mathbf{i}$, and $< 0,1 >$, which is denoted $\mathbf{j}$. These are known as the **standard basis vectors** for the x,y plane.

In three-dimensional space, there are *three* unit vectors that are special. They are $< 1,0,0 >$, which is denoted $\mathbf{i}$, $< 0,1,0 >$, which is denoted $\mathbf{j}$, and $< 0,0,1 >$, which is denoted $\mathbf{k}$. These are known as the **standard basis vectors** for x,y,z space.

In either two or three dimensions, every vector can be expressed as a linear combination of the standard basis vectors:

- If $\mathbf{a} = < a_1, a_2 >$, then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$.
- If $\mathbf{a} = < a_1, a_2, a_3 >$, then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

Example 23: $< 3, -8 > = 3\mathbf{i} - 8\mathbf{j}$, and $< -6, -2, 5 > = -6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$.

In two-dimensional space, $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$, while in three-dimensional space, $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

When a vector is expressed as a linear combination of the standard basis vectors, we say it is expressed in **standard basis form**. (Thus, we now have two notations for symbolically representing any vector, *component form* and *standard basis form*.)