3.6 Derivatives of Logarithmic Functions

In this section we use implicit differentiation to find the derivative of the logarithmic functions $\boldsymbol{y = \log_b x \ and \ y = \ln(x)}$.

To prove the derivatives of the functions above, we need the following derivative.

$$\frac{d}{dx}[b^x] = b^x \ln(b)$$

Let's find the derivative of $y = \log_b x$.

Using the exponential definition we can write $\boldsymbol{y = \log_b x}$ as $\boldsymbol{b^y = x}$.

$$\frac{d}{dx}[b^y] = \frac{d}{dx}[y]$$

$$b^y \ln(b)\, y' = 1 \qquad \text{solve for } y'$$

$$y' = \frac{1}{b^y \ln(b)} \qquad (b^y = x) \text{ substitute}$$

$$y' = \frac{1}{x \ln(b)}$$

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln(b)}$$

If we let $b = e$, for $y = \ln(x) \ \rightarrow \ e^y = x$. Take the derivative of $e^y = x$ implicitly with respect to x.

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y \cdot \ln e \cdot y' = 1 \qquad \text{Solve for } y'.$$

$$y' = \frac{1}{e^y} \qquad y = \ln(x) \text{ – substitute}$$

$$y' = \frac{1}{e^{\ln x}}$$

$$\boldsymbol{y' = \frac{1}{x}} \text{ therefore } \frac{d}{dx}[\ln x] = \frac{1}{x}$$

In general, using the chain rule we get: $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}$ or $\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}$

Example: Find $\frac{d}{dx}[\ln(\cos(x))]$

$$\frac{d}{dx}[\ln(\cos(x))] = \frac{1}{\cos(x)} \cdot -\sin(x)$$

$$= -\frac{\sin(x)}{\cos(x)}$$

$$= -\tan(x)$$

Example: Differentiate $f(x) = \log_{10}(1 + \cos(x))$ Using $\frac{d}{dx}[\ln f(x)] = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}$

$$f'(x) = \frac{1}{(1 + \cos(x)) \cdot \ln(10)} \cdot -\sin(x) = \frac{-\sin(x)}{\ln(10)(1 + \cos(x))}$$

Example: Given: $f(x) = (\ln x)^2 \cdot \sin(x)$, find $f'(x)$ (This will use the product rule, natural log rule & chain rule.)

Let $(\ln x)^2$ = the first part and $\sin(x)$ = the second part.

d-first $= 2\frac{\ln x}{x}$ and d-second $= \cos(x)$

$f'(x)$ = first · d-second + second · d-first

$$\boldsymbol{f'(x) = (\ln x)^2 \cos(x) + \sin(x)\left(2\frac{\ln x}{x}\right)}$$

Example: Differentiate $y = \tan[\ln(ax+b)]$ Using substitution may make this problem easier. (The a & b are constants.) Let u = ln(ax + b). Remember that the derivative of tan(u) = sec²(u)·u' and u' $= \frac{a}{ax+b}$

$$\boldsymbol{y' = sec^2(\ln(ax+b)) \cdot \frac{a}{ax+b}}$$

Logarithmic Differentiation

The calculations of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking the logarithm of the entire equation. This method is called **Logarithmic Differentiation.**

<u>**Steps in Logarithmic Differentiation**</u>

1. Take the natural logarithm of both sides of an equation and use the laws of logarithms to simplify
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

Example: Differentiate the following using logarithmic differentiation.

a) $y = x^x$

b) $y = (\cos x)^x$

a) $y = x^x$

$\ln(y) = \ln(x^x)$

$\ln(y) = x\ln(x)$ Implicit Diff.

$\frac{1}{y}y' = \ln(x) + x \cdot \frac{1}{x}$

$\frac{1}{y}y' = \ln(x) + 1$ Solve for y'

$y' = y(\ln(x) + 1)$ Remember $y = x^x$

$\boldsymbol{y' = x^x(\ln x + 1)}$

b) $y = (\cos x)^x$

$\ln(y) = x\ln(\cos x)$ Implicit Diff.

$\frac{1}{y}y' = x\left(\frac{1}{\cos x}(-\sin x)\right) + \ln(\cos x)$ Solve for y'

$y' = \frac{-x\sin x}{\cos x} + \ln(\cos x)$

$y' = (\cos x)^x\left(\frac{-x\sin x}{\cos x} + \ln(\cos x)\right)$ Remember $y = (\cos x)^x$

$\boldsymbol{y' = (\cos x)^x(-x\tan(x) + \ln(\cos x))}$

c) $y = (\ln(x))^{\cos(x)}$

a) $y = (\ln(x))^{\cos(x)}$

$\ln(y) = \ln\left((\ln(x))^{\cos(x)}\right)$ Use Power rule of logs

$\ln(y) = \cos(x) \cdot \ln(\ln(x))$ Differentiate implicitly with logs and product rule

$\frac{y'}{y} = \cos(x) \cdot \left(\frac{1}{\ln(x)} \cdot \frac{1}{x}\right) + \ln(\ln(x)) \cdot (-\sin(x))$ Solve for y'.

$y' = y\left[\cos(x) \cdot \left(\frac{1}{\ln(x)} \cdot \frac{1}{x}\right) + \ln(\ln(x)) \cdot (-\sin(x))\right]$ substitute $(\ln(x))^{\cos(x)}$ in for y

$y' = (\ln(x))^{\cos(x)}\left[\cos(x)\left(\frac{1}{\ln(x)} \cdot \frac{1}{x}\right) + \ln(\ln(x)) \cdot (-\sin(x))\right]$ simplify

$$\boldsymbol{y' = (\ln(x))^{\cos(x)}\left[\frac{\cos(x)}{x\ln(x)} - \sin(x)\ln(\ln(x))\right]}$$