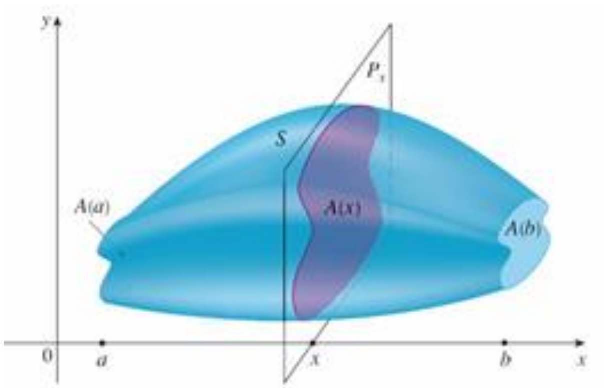
6.2 Volumes

When finding the volume of a solid we have the same problem as we had when finding the areas in the last section.

Consider the following solid, **S.**



We can find the volume of the solid by finding the area of the cross section of ***S*** in the plane ***P_x*** perpendicular to the **x-axis** and passing through point **x** for all **x** in [a, b]. Notice that the corss-section area ***A(x)*** will vary as **x** increases from **a** to **b**. Since ***A(x)*** will vary, lets divide ***S*** into **n** "slabs" of equal width Δx by using the planes $P_{x_1}, P_{x_2}, \dots$ to slice the solid. If we choose sample point x_i^* in $[x_{i-1}, x_i]$, the i^{th} interval, we can approximate the i^{th} slab, S_i by a cylinder with base area $A(x_i^*)$ and height Δx.

The volume of this cylinder is $A(x_i^*)\Delta x$ **…** $V(S_i) \approx A(x_i^*)\Delta x$

Adding the volumes of all of the slabs give us an approximation to the total volume of the solid.

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

The approximation becomes better as $n \to \infty$.

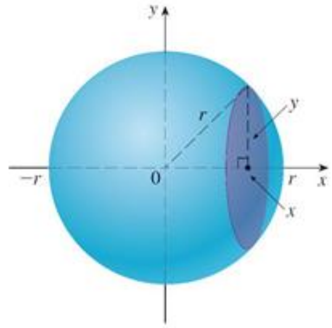
Definition of Volume: Let **S** be a solid that lies between **x = a** and **x = b.** If the cross sectional area of **S** in the plane **P_x**, through **x** and perpendicular to the **x-axis** is **A(x)**, where **A** is a continuous function, the the volume of **S** is:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

It is important to recognize when the area of a moving cross section is changing and when it is not.

Example: Show that the volume of a sphere with radius **r** is $V = \frac{4}{3}\pi r^3$.

Lets draw a diagram of the sphere with the center at the origin.



If we place the sphere so that its center is at the origin, then the plane $\mathbf{P_x}$ intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is $\boldsymbol{y} = \sqrt{\boldsymbol{r^2 - x^2}}$. So the cross-sectional area is
$\mathbf{A(x)} = \boldsymbol{\pi y^2}$ (and since $\boldsymbol{y} = \sqrt{\boldsymbol{r^2 - x^2}}$ we can substitute)
$\mathbf{A(x)} = \boldsymbol{\pi(r^2 - x^2)}$

Using the definition of volume with **a = -r** and **b = r** (limits of integrations), we have

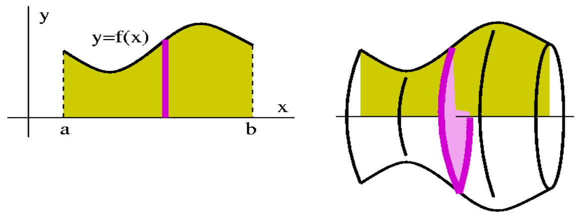
$$V = \int_{-r}^{r} A(x)dx$$
$$= \int_{-r}^{r} \pi(r^2 - x^2)dx$$
$$= 2\pi \int_{0}^{r} (r^2 - x^2)dx$$
$$= 2\pi \left[r^2x - \frac{r^3}{3}\right]_0^r$$
$$= 2\pi \left(r^3 - \frac{r^3}{3}\right)$$
$$= \boldsymbol{\frac{4}{3}\pi r^3}$$

The figure below illustrates the definition of volume when the solid is a sphere with radius r = 1. We know that when r = 1, the volume of a sphere is $\frac{4}{3}\pi(1)^3 \approx 4.18879$.

(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

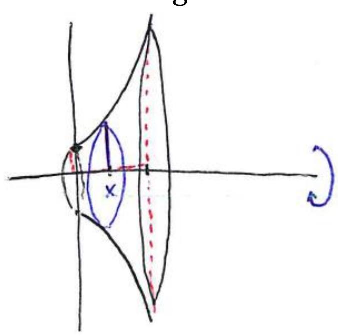
Approximating the volume of a sphere with radius 1

Now we consider a specific type of solid known as a **solid of revolution.** Suppose ***f*** is a continuous function with ***f(x)*** $\geq$ ***0*** on an interval [a, b]. Let **R** be the region bounded by the graph of ***f***, the **x-axis**, and the lines **x = a** and **x = b**. Now revolve **R** around the **x-axis**. As **R** revolves once about the **x-axis**, it sweeps out a 3 – dimensional solid of revolution. The goal is to find the volume of this solid. The figures below show an illustration of this.



Example: Let **R** be the region bounded by the curve **f(x) = (x + 1)²,** the **x-axis**, and the lines **x = 0** and **x = 2.** Find the volume of the solid of revolution obtained by revolving **R** about the **x-axis.**

Below is the figure created by the given information.



A cross-sectional area perpendicular to the x-axis at the point $0 \leq x \leq 2$ is a circular disk whose radius is determined by the function **f(x).**

The cross-sectional area is (using the area of a circle $A = \pi r^2$)
$\mathbf{A(x)} = \boldsymbol{\pi(f(x)^2) = \pi((x+1)^2)^2 = \pi(x+1)^4}$

Integrating those cross-sectional areas between **x = 0** and **x = 2** give the volume of the solid.

$$V = \int_0^2 A(x)dx = \int_0^2 \pi(x+1)^4 dx$$

Let **u** = x + 1, then **du** = dx (x = 0 → u = 1 and x = 2 → u = 3)

$$= \pi \int_1^3 u^4 du = \pi \left[\frac{u^5}{5}\right]_1^3 = \pi \left[\frac{243}{5} - \frac{1}{5}\right] = \frac{\mathbf{242\pi}}{\mathbf{5}}$$

This is called the **disk** method.

A small variation of the method above allows us to compute the volume of more complex solids. Suppose that **R** is the region bounded by the graphs of ***f*** and ***g*** between **x = a** and **x = b**, where ***f(x)*** $\leq$ ***g(x)*** $\leq$ ***0.*** If **R** is revolved about the **x-axis** to generate a solid of revolution, the resulting solid generally has a **hole** through it.

If ***f(x)*** $\leq$ ***g(x)*** $\leq$ ***0***, then ***f(x)*** *is the outer radius,* r_o, and ***g(x)*** is the inner radius, r_i.

The cross section is the area of the entire disk minis the area of the hole. This is called the **washer** method.

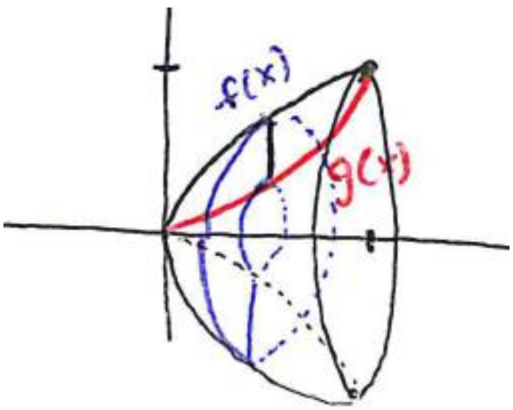
$$A(x) = \pi(r_o^2 - r_i^2) = \pi(f(x)^2 - g(x)^2)$$

Let ***f*** and ***g*** be continuous functions with ***f(x)*** $\leq$ ***g(x)*** $\leq$ ***0*** on [a, b]. Let **R** be the region bounded by **y = f(x)** and **y = g(x)**, and the lines **x = a** and **x = b.** When **R** is revolving about the **x-axis**, the volume of the resulting solid of revolution is:

$$V = \int_a^b \pi(f(x)^2 - g(x)^2)dx$$

Example: The region **R** is bounded by the graphs $f(x) = \sqrt{x}$ and $g(x) = x^2$, between **x = 0** and **x = 1.** What is the volume of the solid that results when revolving **R** about the **x-axis**?

First plot the functions to determine which is greater on the domain.



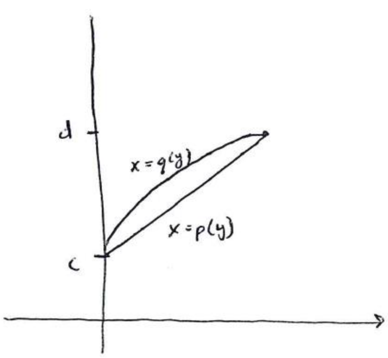
Note: f(x) $\geq$ g(x) on the domain [0, 1]

$A(x) = \pi(f(x)^2 - g(x)^2) = \pi\left(\sqrt{x}^2 - (x^2)^2\right) = \pi(x - x^4)$. The volume of the solid is:

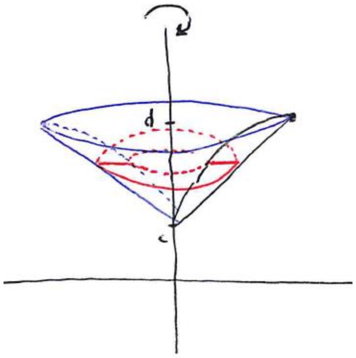
$$V = \int_0^1 \pi(x - x^4)dx = \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3\pi}{10}$$

We can also find volumes of solid by **rotating** about the **y-axis.** The idea is similar to the one of rotating about the x-axis.

Consider a region **R** bounded by the curve **x = p(y)** on the right, the curve **x = q(y)** on the left, and the horizontal lines **y = c** and **y = d.**



The area of the cross section $A(y) = \pi(p(y)^2 - q(y)^2)$, where $c \leq y \leq d$. Notice that now everything is written in terms of **y** making **y** the independent variable and **x** the dependent variable. Now by combining all of the cross-sectional areas of the solid gives us the volume.



Let **p** and **q** be continuous functions with $\boldsymbol{p(y) \geq q(y) \geq 0}$ on [c, d]. Let **R** be the region bounded by **x = p(y), x = q(y),** and the lines **y = c** and **y = d**. When **R** is revolved about the **y-axis**, the volume of the resulting solid of revolution is given by:

$$\boldsymbol{V = \int_c^d \pi(p(y)^2 - q(y)^2)dy}$$

Example: Find the volume of the solid obtained by rotating the region bounded by $\boldsymbol{y = x^3}$, $\boldsymbol{y = 8}$ and **x = 0** about the **y-axis.**

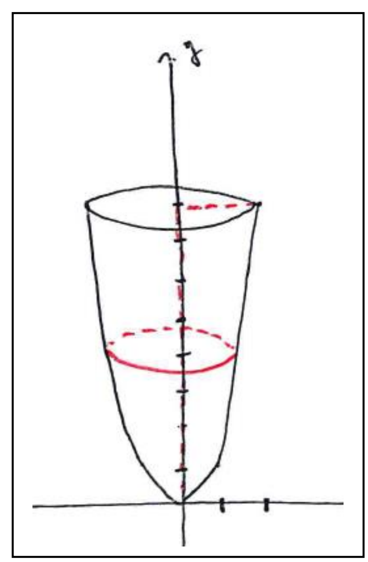
Graph the functions and illustrate the rotation. Since we are revolving about the **y-axis** we need to rewrite the function $\boldsymbol{y = x^3}$ as $\boldsymbol{x = \sqrt[3]{y} = y^{\frac{1}{3}}}$.

$V = \int_0^8 A(y)dy \;=\; \int_0^8 \pi x^2 dy$ (substitue $\boldsymbol{y^{\frac{1}{3}}}$ in for x)

$V = \int_0^8 \pi \left(y^{\frac{1}{3}}\right)^2 dy$

$V = \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^8$

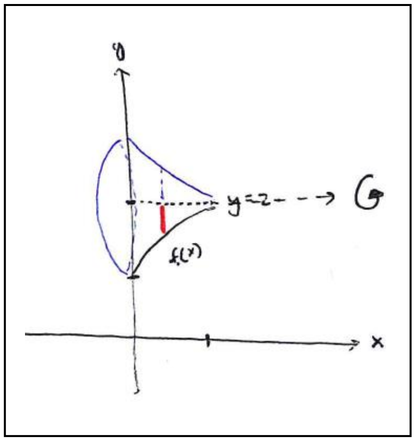
$\boldsymbol{V = \frac{96\pi}{5}}$



Volumes of solids with rotations other than the **x-axis** or **y-axis** can also be found.

Example: Find the volume of the solid generated when a region **R** bounded by the graph $\boldsymbol{f(x) = \sqrt{x} + 1}$ and the line **y = 2** on the interval [0, 1] is revolved about the line **y = 2**

Graph the given information. The radius at any point in **x** would be: r = 2 – f(x)



$r = 2 - (\sqrt{x}+1)$

$r = 2 - \sqrt{x} - 1$

$r = \mathbf{1} - \sqrt{\boldsymbol{x}}$

Therefore, the volume of the solid revolved about $\mathbf{y = 2}$ is

$$\int_0^1 \pi\left(1-\sqrt{x}\right)^2 dx \;=\; \pi\int_0^1\left(1-2\sqrt{x}+x\right)dx \;=\; \boldsymbol{\frac{\pi}{6}}$$