13.3, Part Two) Curvature

1. Introduction to Curvature

In two dimensions, the graph of a linear function, $y = mx + b$, or a constant function, $y = b$, is a *line*, which is by definition *straight* (we can say "straight line," but this is redundant). We say that a line has **zero curvature** at every point.

If a function is nonlinear, then its graph is a *curve*–more precisely, a *nonstraight curve* (we can think of a line as a *straight curve*). It is possible to *measure* the degree to which the graph curves at any point. This measurement is called the graph's **curvature** at that point.

This same concept can be applied to two-dimensional curves that do *not* represent the graph of a function (i.e., which fail the vertical line test), such as circles and ellipses. Likewise, the concept can be applied to curves in three dimensions.

Let us assume a given curve has been parameterized with respect to time, t. Its curvature at the point P_t is denoted $\kappa(t)$. Here we are using the Greek letter named "kappa."

It is possible that a curve can have the same curvature at every point on the curve, i.e., at every instant of time. In other words, $\kappa(t)$ could be constant for every value of t and at every point P_t. When this is the case, we can just write κ in place of $\kappa(t)$, since κ does not vary with t. We describe this situation as **motion with constant curvature**.

This is certainly true for *linear motion*, because $\kappa = 0$ for movement along a line. It is also true for *circular motion*, since a circle has a consistent curvature at all points (although in this case, κ will be a *positive* constant, rather than zero).

On the other hand, it is also possible that a curve may *not* have the same curvature at every point–i.e., different points may have different curvatures. The classic example is an ellipse, which may be thought of as a "squashed" circle. For example, consider the ellipse $\frac{x^2}{36} + \frac{y^2}{4} = 1$, which may be thought of as a vertically compressed circle. Its x intercepts are $(\pm 6, 0)$ and its y intercepts are $(0, \pm 2)$. Clearly, this ellipse has much greater curvature at its x intercepts than it has at its y intercepts.

A circle is the simplest example of a curve with positive curvature, so let us explore the concept of curvature in the context of circles before we apply the concept to more complicated curves, such as ellipses and parabolas and hyperbolas.

Let us review some fundamental principles of circles from basic geometry...

By definition, a **circle** is the set of points in a plane with a fixed distance from a specified reference point. The fixed distance is known as the **radius** of the circle, and the reference point is known as the **center** of the circle. We denote the radius as r (a positive real number). Any line segment whose endpoints lie on the circle and whose midpoint is the center of the circle is known as a **diameter** of the circle. Furthermore, the term "diameter" can also refer to the *length* of any such line segment; in this sense, it is a positive real number and is denoted d. Note that $d = 2r$.

An **arc** of the circle is a subset of the circle consisting of two points on the circle, known as the **endpoints** of the arc, and all points of the circle traversed when moving along the circle from one endpoint to the other in a specified direction. If the endpoints of an arc are endpoints of a diameter, then the arc consists of exactly half the circle and is known as a **semicircle**. If the endpoints of an arc are the same point, then the arc is the full circle.

The length of a circle arc is known as its **arclength**; this is the distance traversed when moving along the arc from one endpoint to the other. The arclength of the full circle is known as the **circumference** of the circle and is denoted C. A circle with radius r has circumference $2\pi r$. In other words, if you start at any point on the circle and travel all the way around, returning to your starting point (i.e., if you complete one full *revolution*), this is the distance you will have traveled.

A **central angle** for a circle is an angle whose vertex is the center of the circle. Any central angle subtends a unique arc of the circle. The arclength of the subtended arc is equal to the radian measure of the angle multiplied by the radius of the circle. We express this by the formula $s = \theta r$, where r is the circle radius, θ is the radian measure of the central angle, and s is the arclength of the subtended arc. Note that the formula $C = 2\pi r$ is a special case of this formula, i.e., it is the case where $\theta = 2\pi$.

A **unit circle** is a circle whose radius is 1. The circumference of a unit circle is 2π. We shall refer to this as a **standard arclength**.

Given any circle, we may ask the question, what is the ratio between a standard arclength and the circumference of the circle? In other words, if you traverse an arc of the circle, traveling a distance of *one* standard arclength, what *fraction* of the circle will you have traversed?

- If the circle has a radius of 1, then its circumference is 2π, so the ratio is $\frac{2\pi}{2\pi} = 1$.
- If the circle has a radius of 2, then its circumference is 4π, so the ratio is $\frac{2\pi}{4\pi} = \frac{1}{2}$.
- If the circle has a radius of 3, then its circumference is 6π, so the ratio is $\frac{2\pi}{6\pi} = \frac{1}{3}$.
- If the circle has a radius of 4, then its circumference is 8π, so the ratio is $\frac{2\pi}{8\pi} = \frac{1}{4}$.
- If the circle has a radius of 5, then its circumference is 10π, so the ratio is $\frac{2\pi}{10\pi} = \frac{1}{5}$.
- If the circle has a radius of $\frac{1}{2}$, then its circumference is π, so the ratio is $\frac{2\pi}{\pi} = 2$.
- If the circle has a radius of $\frac{1}{3}$, then its circumference is $\frac{2}{3}\pi$, so the ratio is $\frac{2\pi}{\frac{2}{3}\pi} = 3$.
- If the circle has a radius of $\frac{2}{3}$, then its circumference is $\frac{4}{3}\pi$, so the ratio is $\frac{2\pi}{\frac{4}{3}\pi} = \frac{3}{2}$.
- If the circle has a radius of $\frac{1}{4}$, then its circumference is $\frac{1}{2}\pi$, so the ratio is $\frac{2\pi}{\frac{1}{2}\pi} = 4$.
- If the circle has a radius of $\frac{3}{4}$, then its circumference is $\frac{3}{2}\pi$, so the ratio is $\frac{2\pi}{\frac{3}{2}\pi} = \frac{4}{3}$.

Note: When the ratio is a proper fraction, this means you have traversed *that fraction* of the full circle, but when the ratio is an improper fraction or whole number, this means you have made *more than one full trip* around the circle. For instance, when the ratio is $3,$ you have made three full trips around the circle; when the ratio is $\frac{3}{2}$, you have made $1\frac{1}{2}$ full trips around the circle.

In more technical language, a “full trip around the circle” is a *revolution*. Thus, we are examining the number of *revolutions per standard arclength*.

We shall *define* the curvature of a circle as this ratio–i.e., as the number of revolutions per standard arclength.

As you can see from the above examples, this ratio simplifies to the *reciprocal* of the circle’s radius. This can be seen algebraically, because $\frac{2\pi}{2\pi r}$ reduces to $\frac{1}{r}$. Thus, for any circle, *the curvature is the reciprocal of the radius*, i.e., $\kappa = \frac{1}{r}$.

2. Curvature For Non-Circular Curves

At this point, we know the curvature of a line (zero) and the curvature of a circle (the reciprocal of its radius). Now let us broaden our analysis to address curves in general.

The key to understanding this topic is the following insight: As a particle moves along a given path, with position function $\mathbf{r}(t)$, there is an intimate relationship between the following three quantities:

- Its speed of motion, $v(t)$.
- Its speed of direction change, $|\mathbf{T}'(t)|$.
- The curvature of its path, $\kappa(t)$.

The speed of direction change, $|\mathbf{T}'(t)|$, depends directly on both the speed of motion, $v(t)$, and on the curvature, $\kappa(t)$. To see how this interaction works, consider the following situations:

First, suppose our particle is moving along the ellipse $\frac{x^2}{36} + \frac{y^2}{4} = 1$ at a constant speed of motion. Its direction of motion, indicated by $\mathbf{T}(t)$, will change relatively quickly when the particle is at a point of high curvature, such as the x intercepts, $(\pm 6, 0)$, and will change relatively slowly when the particle is at a point of low curvature, such as the y intercepts, $(0, \pm 2)$.

Second, suppose our particle is moving along a circle, which has constant curvature. If its speed of motion is relatively high, then it will change direction relatively quickly, but if its speed of motion is relatively low, then it will change direction relatively slowly.

From the above discussion, we would expect that the speed of direction change is *directly proportional* to both the speed of motion and the curvature. This turns out to be true, but we actually find an even *stronger* connection. It turns out that the proportionality factor is exactly 1, which means that the speed of direction change is simply the *product* of the speed of motion and the curvature, i.e., $|\mathbf{T}'(t)| = v(t)\kappa(t)$.

Since we already know how to find $|\mathbf{T}'(t)|$ and $v(t)$, we now have a way of computing $\kappa(t)$, namely, $\kappa(t) = \frac{|\mathbf{T}'(t)|}{v(t)} = v(t)^{-1}|\mathbf{T}'(t)|$.

I have not done a formal proof of this (we'll get to that later), but let's confirm it in the case of a concrete example that's relatively simple. We already know that a circle has constant curvature, equal to the reciprocal of its radius, so we must confirm that $|\mathbf{T}'(t)| = v(t)\frac{1}{r}$, or $v(t) = r|\mathbf{T}'(t)|$.

Suppose a particle is moving about the circle, $x^2 + y^2 = r^2$, with position function $\mathbf{r}(t) = < r\cos t, r\sin t > = r < \cos t, \sin t >$, so its velocity is $\mathbf{v}(t) = r < -\sin t, \cos t >$, and its speed of motion is $v(t) = r$. Thus, the unit tangent vector is $\mathbf{T}(t) = < -\sin t, \cos t >$, whose derivative is $\mathbf{T}'(t) = < -\cos t, -\sin t > = - < \cos t, \sin t >$. Consequently, the speed of direction change is $|\mathbf{T}'(t)| = 1$. Since $v(t) = r$ and $|\mathbf{T}'(t)| = 1$, the equation $v(t) = r|\mathbf{T}'(t)|$ is confirmed.

Before we move on, let's take a deeper look at circular motion. Let k be a positive real number. Suppose a particle is moving about the circle, $x^2 + y^2 = r^2$, with position function $\mathbf{r}(t) = < r\cos(kt), r\sin(kt) > = r < \cos(kt), \sin(kt) >$, so its velocity is $\mathbf{v}(t) = r < -k\sin(kt), k\cos(kt) > = rk < -\sin(kt), \cos(kt) >$, and its speed of motion is $v(t) = rk$. Thus, the unit tangent vector is $\mathbf{T}(t) = < -\sin(kt), \cos(kt) >$, whose derivative is $\mathbf{T}'(t) = < -k\cos(kt), -k\sin(kt) > = -k < \cos(kt), \sin(kt) >$. Consequently, the speed of direction change is $|\mathbf{T}'(t)| = k$. Since $v(t) = rk$ and $|\mathbf{T}'(t)| = k$, the equation $v(t) = r|\mathbf{T}'(t)|$ is confirmed. But let's look more deeply into this situation...

Notice that the speed of motion involves both the circle's radius, r, and the coefficient k, whereas the speed of direction change involves only the coefficient k. It makes sense that the speed of motion would involve both r and k. The radius dictates the circle's circumference, $2\pi r$. The particle covers this distance when t varies from 0 to $\frac{2\pi}{k}$. For a given value of k, increasing r would mean covering a greater distance in the same amount of time, which implies faster speed of motion. For a given r, increasing k would mean covering the same distance is a smaller amount of time, which again implies faster speed of motion. Furthermore, it makes sense that the speed of direction change would involve k but not r. Assuming we measure time in seconds, the particle requires $\frac{2\pi}{k}$ seconds to make one full trip around the circle. During this time period, its unit tangent vector will change direction through a total of 360 degrees. The larger the value of k, the smaller this time period, which implies faster speed of direction change. However, the radius of the circle has no impact. It takes the same time to move once around the circle, sweeping out 360 degrees of direction change, regardless of the radius. To see this clearly, consider two particles moving simultaneously around two concentric circles, $x^2 + y^2 = 9$ and $x^2 + y^2 = 25$, with respective position functions $\mathbf{r}_1(t) = 3 < \cos(kt), \sin(kt) >$ and $\mathbf{r}_2(t) = 5 < \cos(kt), \sin(kt) >$. Let t varies from 0 to $\frac{2\pi}{k}$, and watch how the unit tangent vector for each particle changes direction. In fact, the two unit tangent vectors are *identical*, so naturally their changes in direction are identical as well! This confirms that the radius has no impact on the speed of direction change.

The formula $\kappa(t) = \frac{|\mathbf{T}'(t)|}{v(t)} = v(t)^{-1}|\mathbf{T}'(t)|$ holds for all motion, not just circular motion. However, it is not the most convenient formula to apply, because $|\mathbf{T}'(t)|$ can be very difficult to calculate. Fortunately, there are other formulas that are much easier to work with...

The Curvature Theorem: $\kappa(t) = v(t)^{-3}|\mathbf{v}(t) \times \mathbf{a}(t)| = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^3}$, assuming $v(t) \neq 0$.

For brevity, we may write $\kappa = v^{-3}|\mathbf{v} \times \mathbf{a}| = \frac{|\mathbf{v} \times \mathbf{a}|}{v^3}$.

Since this theorem refers to the cross product of velocity and acceleration, it is *directly* applicable only for three-dimensional curves, not for two-dimensional curves. However, it can be applied *indirectly* when dealing with two-dimensional curves, because it gives us the following two corollaries, both of which deal with plane curves...

Corollary 1: For a two-dimensional curve with position function $\mathbf{r}(t) = < x(t), y(t) >$, $\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{v(t)^3} = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}}$.

For brevity, we may write $\kappa = \frac{|x'y'' - x''y'|}{v^3} = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}}$.

Corollary 2: In the x,y plane, if we have the graph of a function $y = f(x)$, then the curvature is $\kappa(x) = |f''(x)|(1 + f'(x)^2)^{-3/2} = \frac{|f''(x)|}{\left(1 + f'(x)^2\right)^{3/2}}$.

For brevity, we may write $\kappa = |f''|(1 + f'^2)^{-3/2} = \frac{|f''|}{\left(1 + f'^2\right)^{3/2}}$

3. Formal Definition and Proofs

Earlier, we asserted that $|\mathbf{T}'(t)| = v(t)\kappa(t)$, but we did not prove this. A formal proof requires a formal definition.

Formally, we define the curvature of a curve at a given point as the magnitude of the rate of change of the unit tangent vector *when the curve is parameterized with respect to arclength*. (The definition stipulates this particular parameterization, because otherwise curvature would not be uniquely determined, since there would be infinitely many parameterizations to choose from.) So our formal defintion is $\tilde{\kappa}(s) = \left|\frac{d}{ds}\tilde{\mathbf{T}}(s)\right|$, where s is the arclength parameter. We write $\tilde{\kappa}$ in place of κ and $\tilde{\mathbf{T}}$ in place of $\mathbf{T}$ to clarify that in this equation, curvature and the unit tangent vector are being viewed as functions of s rather than as functions of t.

For any parameterization of the curve with respect to time, t, we have the arclength function $s = s(t)$, and $s'(t) = v(t)$.

$\kappa(t) = \tilde{\kappa}(s(t)) = \left|\frac{d}{ds}\tilde{\mathbf{T}}(s(t))\right|$, and $\mathbf{T}(t) = \tilde{\mathbf{T}}(s(t))$.

$\mathbf{T}'(t) = \frac{d}{dt}\mathbf{T}(t) = \frac{d}{dt}\widetilde{\mathbf{T}}(s(t)) = \frac{d}{ds}\widetilde{\mathbf{T}}(s)\, s'(t) = v(t)\frac{d}{ds}\widetilde{\mathbf{T}}(s).$

Thus, $|\mathbf{T}'(t)| = \left| v(t)\frac{d}{ds}\widetilde{\mathbf{T}}(s) \right| = v(t)\left| \frac{d}{ds}\widetilde{\mathbf{T}}(s) \right| = v(t)\, \widetilde{\kappa}(s) = v(t)\, \widetilde{\kappa}(s(t)) = v(t)\, \kappa(t).$

Proof of the Curvature Theorem:

Since $|\mathbf{T}'(t)| = v(t)\kappa(t)$, $\kappa(t) = v(t)^{-1}|\mathbf{T}'(t)| = v(t)^{-1}v(t)^{-2}|\mathbf{v}(t) \times \mathbf{a}(t)|$
$= v(t)^{-3}|\mathbf{v}(t) \times \mathbf{a}(t)| = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{v(t)^3}.$

QED.

Proof of the Corollary 2 of the Curvature Theorem:

We rewrite our two-dimensional vectors as three-dimensional vectors by appending 0 as the third component...
$\mathbf{r}(t) = < x(t), y(t) > \to < x(t), y(t), 0 >$
$\mathbf{v}(t) = < x'(t), y'(t) > \to < x'(t), y'(t), 0 >$
$\mathbf{a}(t) = < x''(t), y''(t) > \to < x''(t), y''(t), 0 >$

We still have $v(t) = (x'(t)^2 + y'(t)^2)^{1/2}$.

$\mathbf{v}(t) \times \mathbf{a}(t) = 0\mathbf{i} + 0\mathbf{j} + (x'(t)y''(t) - x''(t)y'(t))\mathbf{k} =$
$(x'(t)y''(t) - x''(t)y'(t))\mathbf{k}$

$|\mathbf{v}(t) \times \mathbf{a}(t)| = |x'(t)y''(t) - x''(t)y'(t)|$

Thus, by the Curvature Theorem, $\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{v(t)^3} = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}}.$

QED.

Proof of the Corollary 2 of the Curvature Theorem:

Suppose we have $y = f(x)$. Let $t = x$, and let $\mathbf{r}(t) = < t, f(t), 0 >$.
So $\mathbf{v}(t) = < 1, f'(t), 0 >$, and $\mathbf{a}(t) = < 0, f''(t), 0 >$
$v(t) = \sqrt{1 + f'(t)^2}$, and $v(t)^{-3} = (1 + f'(t)^2)^{-3/2}$

$$\mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = \begin{vmatrix} f'(t) & 0 \\ f''(t) & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & f'(t) \\ 0 & f''(t) \end{vmatrix}\mathbf{k} =$$
$0\mathbf{i} - 0\mathbf{j} + f''(t)\mathbf{k} = < 0, 0, f''(t) >$

$|\mathbf{v}(t) \times \mathbf{a}(t)| = |f''(t)|$

Thus, $\kappa(t) = v(t)^{-3}|\mathbf{v}(t) \times \mathbf{a}(t)| = (1 + f'(t)^2)^{-3/2}|f''(t)|$

Replacing t with $x,$ we obtain our desired result.

Q.E.D.