3.2 The Product and Quotient Rules

In this section we will learn how to derive a function that is composed of ***f(x)*** and ***g(x)*** by being multiplied or divided by one another. For example, if ***h(x) = f(x) · g(x)*** or if ***h(x) = f(x) ÷ g(x),*** then what is ***h'(x)***?

NOTE: $(\boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x}))' \neq \boldsymbol{f}'(\boldsymbol{x}) \cdot \boldsymbol{g}'(\boldsymbol{x})$

So let's say we have ***h(x) = f(x)· g(x)*** and we want to find ***h'(x)***.

If ***f(x)*** and ***g(x)*** are both differentiable then we can find ***h'(x)*** by using the definition of the derivative. This proof is found on pages 183 – 184 in your textbook.

The Product Rule: If ***f*** and ***g*** are both differentiable, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$ ... or another way of writing it would be

$$[f(x) \cdot g(x)]' = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Example: Find the following derivatives using The Product Rule.

a) $f(x) = \sqrt{x} \cdot e^x$

b) $f(x) = (1 - e^x)(x + e^x)$

a) Let $f(x) = g(x)h(x)$ where $g(x) = \sqrt{x}$ and $h(x) = e^x$ then

$f'(x) = g(x)h'(x) + h(x)g'(x)$ From previous work we know $g'(x) = \frac{1}{2\sqrt{x}}$ and $h'(x) = e^x$ so using substitution into the Product Rule we get

$$f'(x) = (\sqrt{x})(e^x) + (e^x)\left(\frac{1}{2\sqrt{x}}\right)$$

$$\boldsymbol{f'(x) = e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)}$$

b) Let $f(x) = g(x)h(x)$ where $g(x) = (1 - e^x)$ and $h(x) = (x + e^x)$

$g'(x) = 0 - e^x = -e^x$ and $h'(x) = 1 + e^x$ so now substitute into the product rule

$$f'(x) = (1 - e^x)(1 + e^x) + (x + e^x)(-e^x)$$

$$f'(x) = 1 - e^{2x} - e^{2x} - xe^x$$

$$\boldsymbol{f'(x) = 1 - 2e^{2x} - xe^x}$$

The Quotient Rule: If ***f*** and ***g*** are differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$ this could also we written as $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g\prime(x)}{(g'x)^2}$

Example: Find the derivatives of the following functions using the Quotient Rule.

a) $f(x) = \frac{\sqrt{x}}{2+x}$ Let $f(x) = \frac{g(x)}{h(x)}$ where $g(x) = \sqrt{x}$ and $h(x) = 2 + x$ then $g'(x) = \frac{1}{2\sqrt{x}}$, $h'(x) = 1$

Using the quotient rule and substituting we have: $f'(x) = \frac{(2+x)\left(\frac{1}{2\sqrt{x}}\right)-(\sqrt{x})(1)}{(2+x)^2} = \frac{\frac{2+x}{2\sqrt{x}}-\sqrt{x}}{(2+x)^2}$

b) $f(x) = \frac{g(x)}{x}$ $f'(x) = \frac{x\cdot g'(x)-g(x)\cdot 1}{x^2}$ $f'(x) = \frac{xg'(x)-g(x)}{x^2}$

Example: Differentiate the following function.

$f(x) = \frac{4+x}{x\cdot e^x}$ Let $g(x) = 4 + x$ therefore $g'(x) = 1$ and $h(x) = x \cdot e^x$ using the product rule we get

$h'(x) = xe^x + e^x$

$f'(x) = \frac{xe^x(1)-(4+x)(xe^x+e^x)}{(xe^x)^2}$