Derivatives of Implicit Functions

Part One

Say we have a function $z = f(x,y)$, whose graph in x,y,z space is a surface, S, which passes the Vertical Line Test. Assume the domain of f is the entire x,y plane.

Let $y = g(x)$ be a function whose graph in the x,y plane is a curve, C_1, which passes the Vertical Line Test. Assume the domain of g is all x. Any point on C_1 may be referred to as (x,y) or as $(x,g(x))$.

For any point (x,y) or $(x,g(x))$ on curve C_1, the corresponding point on surface S has z coordinate $f(x,y)$ or $f(x,g(x))$. We now define a function $z = h(x)$ by the rule $h(x) = f(x,g(x))$. The graph of this function in the x,z plane is a curve, C_2, which passes the Vertical Line Test. The domain of h is all x.

For the function h, we can draw a tree diagram where z depends on x and y (accordining to the function f) and where y depends on x (according to the function g). By the Chain Rule, $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$. In other words, $h'(x) = f_x(x,y) + f_y(x,y)g'(x)$.

Note that $\frac{dz}{dx}$ or $h'(x)$ is the slope of the curve C_2 in the x,z plane, whereas $\frac{dy}{dx}$ or $g'(x)$ is the slope of curve C_1 in the x,y plane.

For example, suppose $z = f(x,y) = x^2 + y^2$, whose graph is a circular paraboloid. Let $y = g(x) = \sin x$. Let $z = h(x) = x^2 + \sin^2 x$. On the one hand, we can find $h'(x)$ directly: $h'(x) = 2x + 2\sin x \cos x$. On the other hand, we can use the Chain Rule. $\frac{\partial z}{\partial x} = 2x$. $\frac{\partial z}{\partial y} = 2y$. $\frac{dy}{dx} = \cos x$. So $\frac{dz}{dx} = 2x + 2y\cos x$. Substituting $\sin x$ in place of y gives us $2x + 2\sin x \cos x$.

Here is a more complicated example: $z = f(x,y) = x^4 + y^5 + 6x^3y^4$. Let $y = g(x) = x^2$. Let $z = h(x) = x^4 + (x^2)^5 + 6x^3(x^2)^4 = x^4 + x^{10} + 6x^{11}$. On the one hand, we can find $h'(x)$ directly: $h'(x) = 4x^3 + 10x^9 + 66x^{10}$. On the other hand, we can use the Chain Rule.
$\frac{\partial z}{\partial x} = 4x^3 + 18x^2y^4$. $\frac{\partial z}{\partial y} = 5y^4 + 24x^3y^3$. $\frac{dy}{dx} = 2x$. So
$\frac{dz}{dx} = (4x^3 + 18x^2y^4) + (5y^4 + 24x^3y^3)(2x) = 4x^3 + 18x^2y^4 + 10xy^4 + 48x^4y^3$.
Substituting x^2 in place of y gives us $4x^3 + 18x^2(x^2)^4 + 10x(x^2)^4 + 48x^4(x^2)^3 =$
$4x^3 + 18x^{10} + 10x^9 + 48x^{10} = 4x^3 + 10x^9 + 66x^{10}$.

Now suppose the curve C_1 is a level curve for the function f, i.e., C_1 consists of all points (x,y) such that $z = f(x,y) = k$, where k is some constant. In this scenario, C_1 may or may not pass the Vertical Line Test. If it does, then we still have y as an explicit function of x, but if it does not, then we have y as an *implicit* function of x. In either case, we will continue to write $y = g(x)$ for points on curve C_1, but bear in mind that now g may be implicit rather than explicit. The domain of g might no longer be all x. Let D_g denote the domain of g. Once again, let $z = h(x) = f(x,g(x))$. The domain of h is the same as the domain of g, i.e., D_g. For any $x \in D_g$, the point $(x,g(x))$ lies on curve C_1, so $z = h(x) = f(x,g(x)) = k$. In other words, h is a constant function (so curve C_2 is a horizontal line). So $h'(x) = 0$ for all $x \in D_g$. But the Chain Rule is still applicable, so we still have $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$, or $h'(x) = f_x(x,y) + f_y(x,y)g'(x)$. We set this equal to 0 and solve for $\frac{dy}{dx}$ or $g'(x)$. The result is $\frac{dy}{dx} = -\frac{\partial z}{\partial x} \div \frac{\partial z}{\partial y}$, or $g'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$.

Again, suppose $z = f(x,y) = x^2 + y^2$, and consider the level curve $x^2 + y^2 = 9$. This is a circle in the x,y plane centered at the origin with radius 3. This curve fails the Vertical Line Test, but it gives us y as an implicit function of x, namely, $y = \pm\sqrt{9 - x^2}$. In the first or second quadrant, we have $y = g(x) = \sqrt{9 - x^2}$. The domain of g is $[-3,3]$.

Let $z = h(x) = f(x, \sqrt{9-x^2}) = x^2 + \sqrt{9-x^2}^{\,2} = x^2 + 9 - x^2 = 9$. $h'(x) = 0$ for all $x \in [-3,3]$. Since $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y$, it follows that $\frac{dy}{dx} = -(2x) \div (2y) = -\frac{x}{y}$. If we are in the first or second quadrant, we may substitute $\sqrt{9-x^2}$ in place of y, giving us $\frac{dy}{dx} = -\frac{x}{\sqrt{9-x^2}}$. This same result could have been obtained directly: Since $y = \sqrt{9-x^2} = (9-x^2)^{1/2}$, $\frac{dy}{dx} = \frac{1}{2}(9-x^2)^{-1/2}(-2x) = -x(9-x^2)^{-1/2} = -\frac{x}{\sqrt{9-x^2}}$.

Part Two

Say we have a function $w = f(x,y,z)$, whose graph in x,y,z,w space is a hyper-surface, H, which passes the Vertical Line Test. Assume the domain of f is all x,y,z space.

Let $z = g(x,y)$ be a function whose graph in x,y,z space is a surface, S_1, which passes the Vertical Line Test. Assume the domain of g is the entire x,y plane. Any point on S_1 may be referred to as (x,y,z) or as $(x,y,g(x,y))$.

For any point (x,y,z) or $(x,y,g(x,y))$ on surface S_1, the corresponding point on hyper-surface H has w coordinate $f(x,y,z)$ or $f(x,y,g(x,y))$. We now define a function $w = h(x,y)$ by the rule $h(x,y) = f(x,y,g(x,y))$. The graph of this function in x,y,w space is a surface, S_2, which passes the Vertical Line Test. The domain of h is the entire x,y plane.

For the function h, we can draw a tree diagram where w depends on x, y, and z (accordining to the function f) and where z depends on x and y (according to the function g). By the Chain Rule:

- $h_x(x,y) = f_x(x,y,z) + f_z(x,y,z)g_x(x,y)$
- $h_y(x,y) = f_y(x,y,z) + f_z(x,y,z)g_y(x,y)$

If we want to write these equations in Leibniz notation, we must be careful to avoid ambiguity. First, here are the unambiguous notations:

- For $g_x(x,y)$, we can write either $\frac{\partial g}{\partial x}$ or $\frac{\partial z}{\partial x}$.
- For $g_y(x,y)$, we can write either $\frac{\partial g}{\partial y}$ or $\frac{\partial z}{\partial y}$.
- For $f_z(x,y,z)$, we can write either $\frac{\partial f}{\partial z}$ or $\frac{\partial w}{\partial z}$.

On the other hand, here are the potentially problematic notations:

- $f_x(x,y,z)$ could be written as $\frac{\partial f}{\partial x}$ or $\frac{\partial w}{\partial x}$.
- $f_y(x,y,z)$ could be written as $\frac{\partial f}{\partial y}$ or $\frac{\partial w}{\partial y}$.
- $h_x(x,y)$ could be written as $\frac{\partial h}{\partial x}$ or $\frac{\partial w}{\partial x}$.
- $h_y(x,y)$ could be written as $\frac{\partial h}{\partial y}$ or $\frac{\partial w}{\partial y}$.

Do you see the problem? The notation $\frac{\partial w}{\partial x}$ could refer to either $f_x(x,y,z)$ or to $h_x(x,y)$, and the notation $\frac{\partial w}{\partial y}$ could refer to either $f_y(x,y,z)$ or to $h_y(x,y)$. To avoid ambiguity, we will use $\frac{\partial w}{\partial x}$ *only* to refer to $f_x(x,y,z)$, and we will use $\frac{\partial w}{\partial y}$ *only* to refer to $f_y(x,y,z)$. We will refer to $h_x(x,y)$ *only* as $\frac{\partial h}{\partial x}$, and we will refer to $h_y(x,y)$ *only* as $\frac{\partial h}{\partial y}$. Thus, we may write the above equations in Leibniz notation as follows:

- $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$
- $\frac{\partial h}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} = \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

Note that $h_x(x,y)$ or $\frac{\partial h}{\partial x}$ is the slope of the surface S_2 in x,y,w space in the direction of $\mathbf{i} = < 1,0 >$, and $h_y(x,y)$ or $\frac{\partial h}{\partial y}$ is the slope of the surface S_2 in x,y,w space in the direction of $\mathbf{j} = < 0,1 >$, whereas $g_x(x,y)$ or $\frac{\partial g}{\partial x}$ is the slope of the surface S_1 in x,y,z space in the direction of $\mathbf{i} = < 1,0 >$, and $g_y(x,y)$ or $\frac{\partial g}{\partial y}$ is the slope of the surface S_1 in x,y,z space in the direction of $\mathbf{j} = < 0,1 >$.

For example, suppose $w = f(x,y,z) = x^2 + y^2 + z^2$. Let $z = g(x,y) = x^2 - y^2$, whose graph is a hyperbolic paraboloid. Let $w = h(x,y) = x^2 + y^2 + (x^2 - y^2)^2 = x^2 + y^2 + x^4 - 2x^2y^2 + y^4$. On the one hand, we can find $h_x(x,y)$ and $h_y(x,y)$ directly:

- $h_x(x,y) = 2x + 4x^3 - 4xy^2$
- $h_y(x,y) = 2y - 4x^2y + 4y^3$

On the other hand, we can use the Chain Rule.
$\frac{\partial f}{\partial x} = 2x$. $\frac{\partial f}{\partial y} = 2y$. $\frac{\partial f}{\partial z} = 2z$. $\frac{\partial g}{\partial x} = 2x$. $\frac{\partial g}{\partial y} = -2y$.
So $\frac{\partial h}{\partial x} = 2x + (2z)(2x) = 2x + 4xz$, and $\frac{\partial h}{\partial y} = 2y + (2z)(-2y) = 2y - 4yz$.
Substituting $x^2 - y^2$ in place of z gives us $\frac{\partial h}{\partial x} = 2x + 4x(x^2 - y^2) = 2x + 4x^3 - 4xy^2$,
and $\frac{\partial h}{\partial y} = 2y - 4y(x^2 - y^2) = 2y - 4x^2y + 4y^3$.

Now suppose the surface S_1 is a level surface for the function f, i.e., S_1 consists of all points (x,y,z) such that $w = f(x,y,z) = k$, where k is some constant. In this scenario, S_1 may or may not pass the Vertical Line Test. If it does, then we still have z as an explicit function of x and y, but if it does not, then we have z as an *implicit* function of x and y. In either case, we will continue to write $z = g(x,y)$ for points on surface S_1, but bear in mind that now g may be implicit rather than explicit. The domain of g might no longer be the entire x,y plane. Let D_g denote the domain of g. Once again, let $w = h(x,y) = f(x,y,g(x,y))$. The domain of h is the same as the domain of g, i.e., D_g. For any $(x,y) \in D_g$, the point $(x,y,g(x,y))$ lies on surface S_1, so $w = h(x,y) = f(x,y,g(x,y)) = k$. In other words, h is a constant function (so surface S_2 is a horizontal plane). So $h_x(x,y) = 0$ and $h_y(x,y) = 0$ for all $(x,y) \in D_g$. But the Chain Rule is still applicable, so we still have $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$ and $\frac{\partial h}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}$. We set these equal to 0 and solve for $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$. The results are:

- $\frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x} \div \frac{\partial f}{\partial z}$, or $g_x(x,y) = -\frac{f_x(x,y,z)}{f_z(x,y,z)}$
- $\frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y} \div \frac{\partial f}{\partial y}$, or $g_y(x,y) = -\frac{f_y(x,y,z)}{f_z(x,y,z)}$

Again, suppose $w = f(x,y,z) = x^2 + y^2 + z^2$, and consider the level surface $x^2 + y^2 + z^2 = 9$. This is a sphere in x,y,z space centered at the origin with radius 3. This curve fails the Vertical Line Test, but it gives us z as an implicit function of x and y, namely, $z = \pm\sqrt{9 - x^2 - y^2}$. In the first through fourth octants, we have $z = g(x,y) = \sqrt{9 - x^2 - y^2}$. The domain of g is closed disk $x^2 + y^2 \leq 9$.

Let $w = h(x,y) = f(x,y,\sqrt{9 - x^2 - y^2}\,) = x^2 + y^2 + \sqrt{9 - x^2 - y^2}^{\,2} = x^2 + y^2 + 9 - x^2 - y^2 = 9$. $h_x(x,y) = 0$ and $h_y(x,y) = 0$ for all (x,y) in the disk $x^2 + y^2 \leq 9$. Since $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = 2z$, it follows that $\frac{\partial g}{\partial x} = -(2x) \div (2z) = -\frac{x}{z}$ and $\frac{\partial g}{\partial y} = -(2y) \div (2z) = -\frac{y}{z}$. If we are in the first through fourth octants, we may substitute $\sqrt{9 - x^2 - y^2}$ in place of z, giving us $\frac{\partial g}{\partial x} = -\frac{x}{\sqrt{9-x^2-y^2}}$ and $\frac{\partial g}{\partial y} = -\frac{y}{\sqrt{9-x^2-y^2}}$. These same results could have been obtained directly:
Since $z = \sqrt{9 - x^2 - y^2} = (9 - x^2 - y^2)^{1/2}$,
$\frac{\partial g}{\partial x} = \frac{1}{2}(9 - x^2 - y^2)^{-1/2}(-2x) = -x(9 - x^2 - y^2)^{-1/2} = -\frac{x}{\sqrt{9-x^2-y^2}}$, and
$\frac{\partial g}{\partial y} = \frac{1}{2}(9 - x^2 - y^2)^{-1/2}(-2y) = -y(9 - x^2 - y^2)^{-1/2} = -\frac{y}{\sqrt{9-x^2-y^2}}$.