7.2 Trigonometric Integrals

The purpose of this section is to develop techniques for evaluating integrals involving trigonometric functions. We start with powers of sine and cosine.

Example: Evaluate:

$$\int cos^5(x)dx$$

Integrals involving **odd** powers of either sine or cosine are easily evaluated by splitting into a single factor or sin(x) or cos(x) times the remaining **even** factor or sin(x) or cos(x) then using the fundamental trigonometric identity of $\sin^2(x) + \cos^2(x) = 1$. (Note: $\cos^5(x) = \cos^4(x) \cdot \cos(x)$)

$$\int cos^5(x)dx = \int cos^4(x) \cdot \cos(x)dx$$

$$= \int (cos^2(x))^2 \cdot \cos(x)\, dx$$

$$= \int (1 - sin^2 x)^2 \cos(x)dx$$

Now integrate with substitution: let **u** = sin(x) and **du** = cos(x)dx

$$= \int (1 - u^2)^2 du \quad = \quad \int (1 - 2u^2 + u^4)du \; = \; u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

Now remember to back substitute: u = sin(x)

$$= \mathbf{\sin(x) - \frac{2}{3} sin^3(x) + \frac{1}{5} sin^5(x) + C}$$

With **even** powers of sine or cosine we use the half-angle identities from trigonometry.

$\sin(x) = \sqrt{\frac{1-\cos(2x)}{2}}$ **or** $sin^2(x) = \frac{1-\cos(2x)}{2}$ $\quad \cos(x) = \sqrt{\frac{1+\cos(2x)}{2}}$ **or** $cos^2(x) = \frac{1+\cos(2x)}{2}$

Example: Evaluate:

$$\int sin^4(x)dx$$

$$\int sin^4(x)dx = \int (sin^2(x))^2 dx = \int \left(\frac{1 - \cos(2x)}{2}\right)^2 dx$$

$$= \int \left(\frac{1 - 2\cos(2x) + cos^2(2x)}{4}\right) dx = \frac{1}{4}\int (1 - 2\cos(2x) + cos^2(2x))$$

Now since $cos^2(x) = \frac{1+\cos(2x)}{2}$ that makes $cos^2(2x) = \frac{1+\cos(4x)}{2}$ so substitute this into the integral.

$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2}\right) dx$$

$$= \frac{1}{4}\int \left(1 - 2cos(2x) + \frac{1}{2} + \frac{\cos(4x)}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right) dx$$

(Using substitution to integrate we get …)

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right]$$

$$= \mathbf{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)}$$

So, if the power of sine or cosine is **odd**, try to write an integrand involving powers of sine or cosine in a form where we have a single sine or cosine factor and the remainder of the expression is in terms of **even** powers of sine or cosine. Then use the fundamental trig identity: $sin^2(x) + cos^2(x) = 1$. If sine or cosine have **even** powers, use the ½ angle identities.

Strategy for Evaluation $\int \boldsymbol{sin^m(x) \cdot cos^n(x) dx}$

(a) If the power of cosine is **odd** (n = 2k+1), save one cosine factor and use $\cos^2(x) = 1 - \sin^2(x)$ to express the remaining factors in terms of sine:

$$\int sin^m x \cdot cos^n x\, dx = \int sin^m x (cos^2 x)^k \cdot \cos(x) dx$$

$$= \int sin^m x (1 - sin^2 x)^k \cos x\, dx$$

Then substitute $\boldsymbol{u} = \sin(x)$.

(b) If the power of sine is **odd** (m=2k+1), save one sine factor and use $\sin^2(x) = 1 - \cos^2(x)$ to Express the remaining factors in terms of cosine:

$$\int sin^{2k+1} x \cdot cos^n x\, dx = \int (sin^2 x)^k \cos x \cdot \sin x\ dx$$

$$= \int (1 - cos^2 x)^k cos^n x \cdot \sin x\, dx$$

Then substitute $\boldsymbol{u} = \cos(x)$

[Note: If both powers of sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of sine and cosine are **even**, use the ½ - angle identities

$$\boldsymbol{sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \qquad cos^2(x) = \frac{1}{2}(1 + \cos(2x))}$$

It is sometimes helpful to use the identity below which is a form of the double angle identity for sine. ($\sin(2x) = 2\sin(x)\cos(x)$)

$$\boldsymbol{\sin(x)\cos(x) = \frac{1}{2}\sin(2x)}$$

Example: Evaluate the integral

$$\int sin^4(x) cos^2(x) dx$$

When both powers are **even** use the ½ - angle identities.

$$\int sin^4(x)cos^2(x)dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 \cdot \left(\frac{1+\cos(2x)}{2}\right)^2 dx$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right) \cdot \left(\frac{1-\cos(2x)}{2}\right) \cdot \left(\frac{1+\cos(2x)}{2}\right) dx$$

$$= \int \frac{1-\cos(2x) - cos^2(2x) + cos^3(2x)}{8} dx$$

$$= \frac{1}{8}\int (1-\cos(2x) - cos^2(2x) + cos^3(2x))\, dx$$

Rewrite the 3rd term of the integrand using a ½ -angle identity. For the last term, which is an **odd** power of cosine, use strategy (a). You can write each term as separate integrals or leave them as one integral.

$$= \frac{1}{8}\int \left(1-\cos(2x) - \frac{1+\cos(4x)}{2}\right) dx + \frac{1}{8}\int cos^2(2x)\cos(2x)dx$$

$$= \frac{1}{8}\int \left(1-\cos(2x) - \frac{1+\cos(4x)}{2}\right) dx + \frac{1}{8}\int \left(1-sin^2(2x)\right)\cos(2x)dx$$

Simplify the first integrand (common denominator) and use **u – substitution** for the second integrand.

Let **u** = sin(2x) ∴ **du** = 2cos(2x)dx ⇒ $\frac{1}{2}$**du** = cos(2x)dx

$$= \frac{1}{8}\int \left(\frac{2-2\cos(2x) - 1 - \cos(4x)}{2}\right) dx + \frac{1}{8}\int (1-u^2)\cdot\frac{1}{2}du$$

$$= \frac{1}{16}\int (1-2\cos(2x) - \cos(4x))dx + \frac{1}{16}\int (1-u^2)du$$

$$= \frac{1}{16}\left[x - \frac{2\sin(2x)}{2} - \frac{\sin(4x)}{4} + C\right] + \frac{1}{16}\left[u - \frac{u^3}{3} + C\right]$$

$$= \frac{1}{16}\left[x - \sin(2x) - \frac{\sin(4x)}{4} + C\right] + \frac{1}{16}\left[\sin(2x) - \frac{sin^3(2x)}{3} + C\right]$$

$$= \frac{1}{16}\left[x - \frac{\sin(4x)}{4} - \frac{sin^3(2x)}{3} + C\right]$$

$$= \mathbf{\frac{1}{16}x - \frac{\sin(4x)}{64} - \frac{sin^3(2x)}{48} + C}$$

Example: Evaluate:

$$\int \sin(x)^3 \cdot cos^{-2}(x)dx$$

$$\int \sin(x)^3 \cdot cos^{-2}(x)dx = \int \sin(x)\cdot sin^2(x)\cdot cos^{-2}(x)dx$$

$$= \int \sin(x)\,(1-cos^2(x))\cdot cos^{-2}(x)dx$$

(Let **u** = cos(x) then **du = -** sin(x)dx ⇒ **- du =** sin(x)dx)

$$= -\int (1-u^2)\cdot u^{-2}du$$

$$= -\int (u^{-2}-1)du \;=\; \int (1-u^{-2})du$$

$$= u + u^{-1} + C = u + \frac{1}{u} + C$$

$$= \cos(x) + \frac{1}{\cos(x)} + C \;=\; \mathbf{\cos(x) + \sec(x) + C}$$

To evaluate an integral such as $\int sin^8(x)dx$ using a method like the one used for $\int sin^4(x)dx$ is tedious. For this reason reduction formulas have been developed. A reduction formula equates an integral involving a power of a function with another integral in which the power is reduced. Below I have given a list of frequently used reduction formulas for trig integrals.

Reduction Formulas for Integrals of Trigonometric Powers:

$$\mathbf{1.)} \int sin^n(x)dx = -\frac{sin^{n-1}(x)\cdot cos(x)}{n} + \frac{n-1}{n}\int sin^{n-2}(x)dx$$

$$\mathbf{2.)} \int cos^n(x)dx = -\frac{cos^{n-1}(x)\cdot sin(x)}{n} + \frac{n-1}{n}\int cos^{n-2}(x)dx$$

$$\mathbf{3.)} \int tan^n(x)dx = \frac{tan^{n-1}(x)}{n-1} - \int tan^{n-2}(x)dx,\ \ n \neq 1$$

$$\mathbf{4.)} \int sec^n(x)dx = \frac{sec^{n-2}(x)\cdot tan(x)}{n-1} + \frac{n-2}{n-1}\int sec^{n-2}(x)dx,\ \ n \neq 1$$

Example: Evaluate using the reduction formulas:

$$\int tan^4(x)dx$$

$$\int tan^4(x)dx = \frac{tan^3(x)}{3} - \int tan^2(x)dx$$

Use the 3rd reduction formula for tangent again.

$$= \frac{tan^3(x)}{3} - \left[\frac{\tan(x)}{1} - \int tan^0(x)dx\right]$$

$$= \mathbf{\frac{tan^3(x)}{3} - \tan(x) + x + C}$$

Note: You could also use $tan^2(x) = sec^2(x) - 1$ by rewriting $tan^4(x) = tan^2(x)\cdot tan^2(x)$

The **odd** powers of tan(x) and sec(x) eventually give $\int \tan(x)\,dx$ and $\int \sec(x)\,dx$.

Theorem: Integrals of **tan(x)**, **cot(x)**, **sec(x)**, and **csc(x)**

$$\int tan(x)dx = -ln|cos(x)| + C = ln|sec(x)| + C$$

$$\int \boldsymbol{sec(x)dx = ln|sec(x) + tan(x)| + C}$$

$$\int \boldsymbol{cot(x)dx = ln|sin(x)| + C}$$

$$\int \boldsymbol{csc(x)dx = -ln|csc(x) + cot(x)| + C}$$

Proof of $\int \boldsymbol{tan(x)dx}$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx$$

Integrate by **u – substitution.** Let **u** = cos(x) then **du = -** sin(x)dx

$$= -\int \frac{1}{u}du = -\ln(u) + C$$

$$= -\ln(\cos(x)) + C$$

Using log properties …

$$= ln(\cos(x)^{-1}) + C \;\; = \mathbf{ln\,|sec(\boldsymbol{x})| + \boldsymbol{C}}$$

We now consider integrals of the form

$$\int tan^m(x)\cdot sec^n(x)dx$$

Strategy for integrating $\int \boldsymbol{tan^m(x)\cdot sec^n(x)dx}$

(a) If the power of secant is **even** (n = 2k, k ≥ 2), save a factor of $sec^2(x)$ and use $sec^2(x) = 1 + tan^2(x)$ to express the remaining factors in terms of tan(x):

$$\int tan^m(x)\cdot sec^{2k}(x)dx = \int tan^m(x)(sec^2(x)^{k-1}sec^2(x)dx$$

$$= \int tan^m\left(1 + tan^2(x)\right)^{k-1} sec^2(x)dx$$

Then substitute **u** = tan(x).

(b) If the power of tangent is **odd** (m = 2k + 1), save a factor of sec(x)tan(x) and use $tan^2(x) = sec^2(x) - 1$ to express the remaining factors in terms of sec(x):

$$\int tan^{2k+1}(x)\cdot sec^n(x)dx = \int (tan^2(x))^k\cdot sec^{n-1}(x)\tan(x)dx$$

$$= \int (sec^2(x) - 1)^k sec^{n-1}(x)\sec(x)\tan(x)\,dx$$

Then substitute **u** = sec(x).

Example: Evaluate the integral

$$\int tan^3(x)\cdot sec^4(x)dx$$

You could either do strategy (a) or (b) from above. Let's use strategy (a) – even power of sec(x), save a factor of $sec^2(x)$.

$$\int tan^3(x)\cdot sec^2(x)\cdot sec^2(x)dx$$

$$\int tan^3(x) \cdot sec^2(x)(tan^2(x)+1)dx$$

Let **u** = tan(x) and **du** = sec²(x)dx

$$\int u^3(u^2+1)du = \int (u^5+u^3)dx = \frac{u^6}{6}+\frac{u^4}{4}+C = \mathbf{\frac{1}{6}tan^6(x)+\frac{1}{4}tan^4(x)+C}$$

Example: Evaluate the integral

$$\int tan^2(x)\sec(x)\,dx$$

Use strategy 3 – write $\tan^2(x)$ in terms of $\sec^2(x)$.

$$\int tan^2(x)\sec(x)\,dx = \int (sec^2(x)-1)\cdot\sec(x)\,dx$$

$$= \int (sec^3(x)-\sec(x))dx$$

$$= \int sec^3(x)dx - \int \sec(x)\,dx$$

(Use reduction formula 4)

$$= \frac{\sec(x)\cdot\tan(x)}{2}+\frac{1}{2}\int \sec(x)\,dx - \int \sec(x)\,dx$$

$$= \frac{\sec(x)\cdot\tan(x)}{2}-\frac{1}{2}\int \sec(x)\,dx$$

$$= \mathbf{\frac{1}{2}\sec(x)\tan(x)-\frac{1}{2}\ln|\sec(x)+\tan(x)|+C}$$

The techniques that were used to integrate problems in the form of $\int \boldsymbol{tan^m(x)\cdot sec^n(x)dx}$ can be used to integrate problems in the form of $\int \boldsymbol{cot^m(x)\cdot csc^n(x)dx}$.