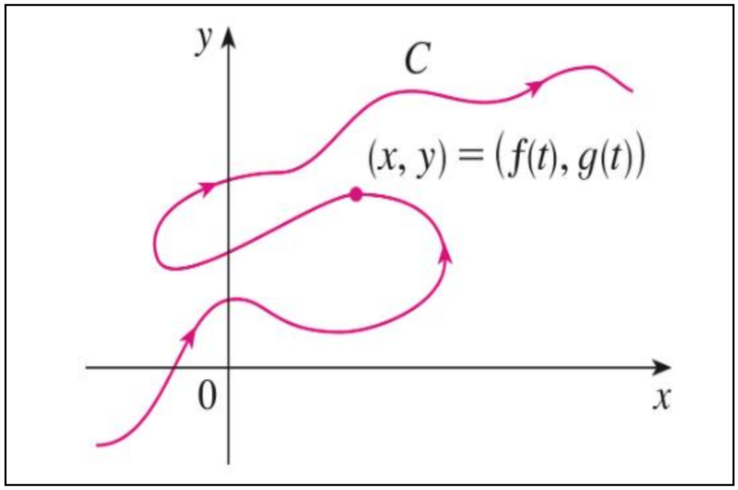
10.1 Curves Defined by Parametric Equations

Parametric equations help us model the trajectories of moving objects. Sometimes it is impossible to describe the trajectory of a moving object with a function of the from **y = f(x)** because the trajectory fails the vertical line test. Consider the following trajectory.



Note that both the x- and y-coordinates of the particle are functions of time so we can write **x = f(t)** and **y = g(t)**. Here **t** is called the parameter.

In addition, each value of **t** gives an ordered Pair (x, y) = (f(t), g(t)). In other words, as **t** Varies so does the point (f(t), g(t)). Connecting Those points gives rise to our parametric curve.

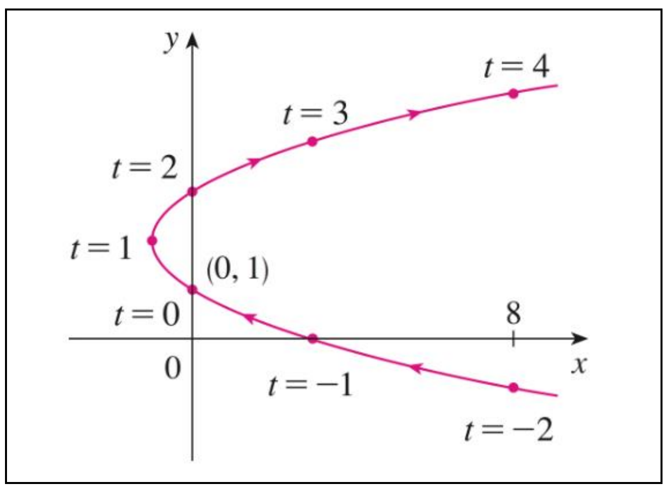
Note: **t** does not always represent time but if it is used in the parametric equations, it probably does.

Example: Graph and analyze the parametric equations $x = t^2 - 2t$ and $y = t + 1$

Create a table for different values of **t**. (Unless we know that **t** represents time, we can have $t < 0$.)

Plot the points.

t	x	y	(x, y)
-2	8	-1	(8, -1)
-1	3	0	(3, 0)
0	0	1	(0, 1)
1	-1	2	(-1, 2)
2	0	3	(0, 3)
3	3	4	(3, 4)
4	8	5	(8, 5)



A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as **t** increases. Notice that consecutive points marked on the curve are at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as **t** increases.

This curve appears to be a parabola. We can confirm this by eliminating the parameter **t** and identifying the equation. Since $y = t + 1$ then we can say $y - 1 = t$ and we can substitute this unto the first equation of $x = t^2 - 2t$.

$$x = (y - 1)^2 - 2(y - 1)$$
$$x = y^2 - 2y + 1 - 2y + 2$$

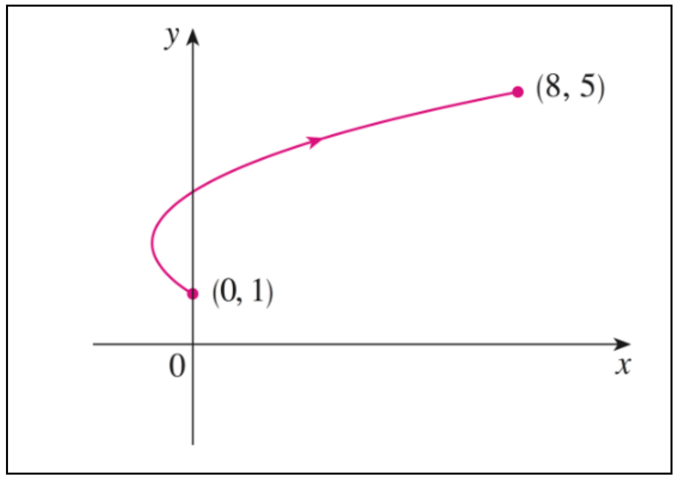
$$\boldsymbol{x = y^2 - 4y + 3}$$

Since $\boldsymbol{x = y^2 - 4y + 3}$ is a parabola opening to the right, we can say that the parametric equations represent a parabola.

No restriction was placed on the parameter **t** in this example so we assumed that **t** could be any real number. **BUT** sometimes we restrict **t** to lie in a finite interval. For instance, consider the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \le t \le 4$$

In this case we would get the graph at the right.



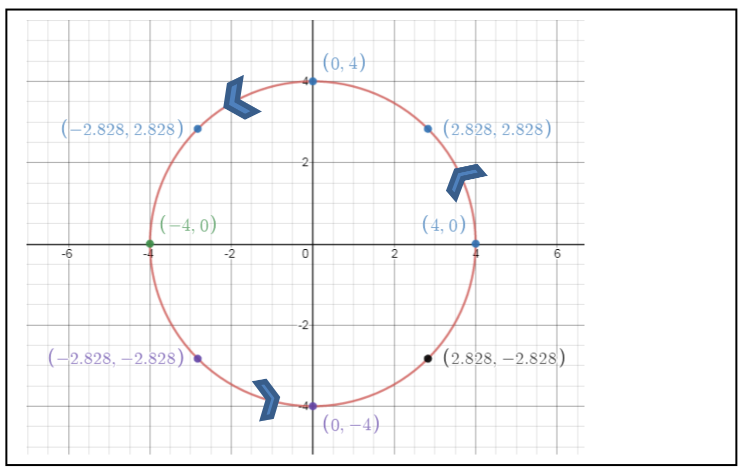
In general, the curve with parametric equations $x = f(t) \quad y = g(t) \quad a \le t \le b$ has an **Initial point** of $(\boldsymbol{f(a)}, \boldsymbol{g(a)})$ and a **terminal point** of $(\boldsymbol{f(b)}, \boldsymbol{g(b)})$.

Example: Graph and analyze the parametric equations

$$x = 4\cos(2\pi t) \qquad y = 4\sin(2\pi t), \qquad for\ 0 \le t \le 1$$

Create a table using different values of **t.**

t	x	y	(x, y)
0	4	0	(4, 0)
$1/8$	$2\sqrt{2}$	$2\sqrt{2}$	$(2\sqrt{2}, 2\sqrt{2})$
$1/4$	0	4	(0,4)
$3/8$	$-2\sqrt{2}$	$2\sqrt{2}$	$(-2\sqrt{2}, 2\sqrt{2})$
$1/2$	-4	0	(-4, 0)
$3/4$	0	-4	(0, -4)
1	4	0	(4, 0)



The result appears to be a circle with radius 4. To identify the curve we can eliminate the parameter **t**.

$$\begin{aligned} x^2 + y^2 &= (4\cos(2\pi t))^2 + (4\sin(2\pi t))^2 \\ &= 16cos^2(2\pi t) + 16sin^2(2\pi t) \\ &= 16[cos^2(2\pi t) + sin^2(2\pi t)] \\ &= 16[1] \\ x^2 + y^2 &= 16 \end{aligned}$$

The graph of the parametric equations is a circle $x^2 + y^2 = 16$.

Example: Consider the parametric equations

$$x = -2 + 3t \qquad y = 4 - 6t \qquad for -\infty \leq t \leq \infty$$

Find the slope-intercept form of the line.

To solve this problem solve one of the parametric equations for **t** and substitute this into the other parametric equation. (In other words, eliminate the parameter **t**.)

$$t = \frac{x+2}{3}$$

$$y = 4 - 6\left(\frac{x+2}{3}\right)$$

$$y = 4 - 2(x+2)$$

$$y = 4 - 2x - 4$$

$$\boldsymbol{y = -2x}$$

It doesn't matter which parametric equation you rewrite. To work this problem starting with the 2nd parametric equation we would get:

$$t = \frac{y-4}{-6}$$

$$x = -2 + 3\left(\frac{y-4}{-6}\right)$$

$$x = -2 + \left(\frac{y-4}{-2}\right)$$

$$x = -2 - \frac{1}{2}y + 2$$

$$x = -\frac{1}{2}y$$

$$\boldsymbol{y = -2x}$$

As you can see we get the same result. However, careful selection of which parametric equation you solve for **t** can result is less complicated work.