Section 7.5

Strategies for Integration

In this section we will practice all of the integration techniques that we have learned thus far. As you have seen, integration is much more difficult that derivative but we will try to analyze a problem and determine what strategies will be useful to calculate the answer. To start, you must know/learn the following table. This table is found on page 503 of your textbook.

TABLE OF INTEGRATION FORMULAS Constants of integration have been omitted.

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$	**2.** $\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert$
3. $\int e^x\,dx = e^x$	**4.** $\int a^x\,dx = \dfrac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$	**6.** $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$	**8.** $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$	**10.** $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert$	**12.** $\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert$
13. $\int \tan x\,dx = \ln\lvert \sec x\rvert$	**14.** $\int \cot x\,dx = \ln\lvert \sin x\rvert$
15. $\int \sinh x\,dx = \cosh x$	**16.** $\int \cosh x\,dx = \sinh x$
17. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	**18.** $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$
***19.** $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\ln\left\lvert \dfrac{x-a}{x+a}\right\rvert$	***20.** $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert$

After you learn the formulas in the table above, if you don't see a clear method to integrate you can try to following four-step strategy.

<u>Strategy</u>

1. Simplify the integrand if possible
2. Look for an obvious substitution
3. Classify the integrand according to its form:
   a. Trigonometric Functions: Use the substitutions recommended in Sections 7.2 and 7.3.
   b. Rational Functions: Partial Fractions recommended in Section 7.4.
   c. Integration by Parts: If f(x) is a product of a power of x (or a polynomial) and a transcendental function (such as trigonometric, exponential, or logarithmic), then try integration by parts. (L-I-A-T-E)
   d. Radicals: Use Trigonometric substitution or rationalizing substitution described in Sections 7.3 and 7.4 respectively.
4. Try again.

   Notice that sometimes it is necessary to use multiple methods of integration when completing one integral.

Example: Evaluate $\int \frac{x}{\sqrt{3-x^4}}dx$

Let $u = x^2$ then $du = 2xdx$ or $\frac{1}{2}du = xdx$

$$\int \frac{x}{\sqrt{3-x^4}}dx = \frac{1}{2}\int \frac{du}{\sqrt{3-u^2}} \quad \textit{now use trig substitution}$$

Let $u = \sqrt{3}\sin(\theta)$ *then* $du = \sqrt{3}\cos(\theta)\,d\theta$

$$\frac{1}{2}\int \frac{\sqrt{3}\cos(\theta)\,d\theta}{\sqrt{3-\left(\sqrt{3}\sin(\theta)\right)^2}} = \frac{1}{2}\int \frac{\sqrt{3}\cos(\theta)\,d\theta}{\sqrt{3(1-sin^2(\theta))}}$$

$$\frac{1}{2}\int \frac{\sqrt{3}\cos(\theta)d\theta}{\sqrt{3}\cos(\theta)} = \frac{1}{2}\int 1\,d\theta = \frac{1}{2}[\theta] + C$$

Remember $u = \sqrt{3}\sin(\theta) \;\rightarrow\; \frac{u}{\sqrt{3}} = \sin(\theta) \;\rightarrow \sin^{-1}\frac{u}{\sqrt{3}} = \theta$

So now we get $\int \frac{x}{\sqrt{3-x^4}}dx = \frac{1}{2}\sin^{-1}\left(\frac{u}{\sqrt{3}}\right) + c$

Also $u = x^2$, therefore

$$\int \frac{x}{\sqrt{3-x^4}}dx = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} + C$$

Example: Evaluate $\int \frac{x-1}{x^2-4x-5}dx$

Perform partial fraction decomposition on $\frac{x-1}{x^2-4x-5}$ $\quad \frac{x-1}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1}$ ... solve for A and B.

$A = \frac{2}{3}$ and $B = \frac{1}{3}$ thus

$$\int \frac{x-1}{x^2-4x-5}dx = \int \frac{2/3}{x-5} + \frac{1/3}{x+1}dx = \frac{2}{3}\ln(x-5) + \frac{1}{3}\ln(x+1) + C$$

Example: Evaluate $\int \frac{x-1}{x^2-4x+5}dx$ Notice that you cannot factor the denominator therefore complete the square. We can rewrite x^2-4x+5 as $(x-2)^2+1$ and rewrite $x-1$ as $(x-2)+1$

$$\int \frac{x-1}{x^2-4x+5}dx = \int \frac{(x-2)+1}{(x-2)^2+1}dx$$

Let $u = x-2 \rightarrow du = dx$

$$\int \frac{u+1}{u^2+1}du = \int \frac{u}{u^2+1}du + \int \frac{1}{u^2+1}du = \frac{1}{2}\ln(u^2+1) + \tan^{-1}(u) + C$$

Now remember to back substitute $x-2$ for u

$$\frac{1}{2}\ln((x-2)^2+1) + \tan^{-1}(x-2) + C = \frac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$$

Example: Evaluate $\int \frac{\sqrt{2x-1}}{2x+3}dx$ Let $u = \sqrt{2x-1}$ then $u^2 = 2x-1$ differentiate both side of the equation $2udu = 2dx \rightarrow udu = dx$. Since $u^2 = 2x-1 \rightarrow u^2+4 = 2x-1+4 = 2x+3$

$$\int \frac{\sqrt{2x-1}}{2x+3}dx = \int \frac{u}{u^2+4}udu = \int \frac{u^2}{u^2+4}du$$

$$\left(\textit{Now doing lots of algebra: } \frac{u^2}{u^2+4} = \frac{u^2+4-4}{u^2+4} = \frac{u^2+4}{u^2+4} + \frac{-4}{u^2+4} = 1 - \frac{4}{u^2+4}\right)$$

$$\int 1 - \frac{4}{u^2+4}dx = \int 1\,dx - \int \frac{4}{u^2+4}dx$$

Let's concentrate on the 2nd integral. From #17 on the 1st page we have $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Therefore,

$$\int \frac{\sqrt{2x-1}}{2x+3}dx = u - 4\left(\frac{1}{2}\tan^{-1}\frac{u}{2}\right) + C$$

Remember to back substitute $\sqrt{2x-1}$ for u.

$$\int \frac{\sqrt{2x-1}}{2x+3}dx = \sqrt{2x-1} + 2\tan^{-1}\left(\frac{\sqrt{2x-1}}{2}\right) + C$$

Example: Evaluate $\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx = \int \frac{1}{\left((1-x^2)^{\frac{1}{2}}\right)^3} dx = \int \frac{1}{\left(\sqrt{1-x^2}\right)^3} dx$$

Use Trig. Substitution. Let $x = \sin(\theta)$ thus $dx = \cos(\theta)d\theta$

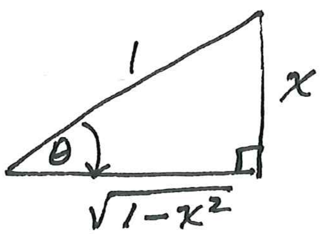
$$\int \frac{1}{\left(\sqrt{1-x^2}\right)^3} dx = \int \frac{\cos(\theta)d\theta}{\left(\sqrt{1-sin^2\theta}\right)^3} = \int \frac{\cos(\theta)\, d\theta}{\left(\sqrt{cos^2\theta}\right)^3} = \int \frac{1}{\cos^2(\theta)} d\theta = \int sec^2(\theta)d\theta = \tan(\theta) + C$$

Using right triangle trigonometry, we get $\tan(\theta) = \frac{x}{\sqrt{1-x^2}}$ therefore

$\sqrt{1-x^2} \rightarrow$ type 1

$x = 1 \sin\theta$

$\frac{x}{1} = \sin\theta$



$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\int \frac{1}{(1-x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{1-x^2}} + C$$