15.3.1) The Polar Coordinate System

In any given plane, there are infinitely many points. The *Cartesian Coordinate System* is a way of associating each point in a given plane with an ordered pair of real numbers, (x,y). When this system has been imposed upon a plane, the plane is then known as a *Cartesian plane*, or *x,y plane*. This correspondence between points and ordered pairs is a *one-to-one correspondence*: Each point in the plane corresponds to a unique ordered pair of real numbers, and each ordered pair of real numbers corresponds to a unique point in the plane. The *entire* plane is represented by the set of *all* ordered pairs of real numbers, $\{(x,y) \mid x \in (-\infty,\infty), y \in (-\infty,\infty)\}$.

An alternative to the Cartesian Coordinate System is the *Polar Coordinate System*, which likewise associates each point in a given plane with an ordered pair of real numbers, (r,Θ). When this system has been imposed upon a plane, the plane is then known as a *polar plane*, or r,Θ plane. This correspondence between points and ordered pairs is *not* a one-to-one correspondence: Each ordered pair of real numbers corresponds to a unique point in the plane, but each point in the plane corresponds to *infinitely many* different ordered pairs of real numbers. In other words, every point has infinitely many different polar coordinates. If a point has polar coordinates (r,Θ), then it also has polar coordinates $(r,\Theta+2n\pi)$ and $(-r,\Theta+(2n+1)\pi)$, where n is any integer. The pole itself has coordinates $(0,\Theta)$, where Θ is any real number.

A given plane can be coordinatized simultaneously with both the Cartesian Coordinate System and the Polar Coordinate System. The standard way of superimposing the two systems is to let the origin of the Cartesian system be the pole of the polar system, and to let the positive x axis be the polar axis. We then have the following equations:

1. $x = r \cos\Theta$,
2. $y = r \sin\Theta$,
3. $r^2 = x^2 + y^2$,
4. $\tan \Theta = y/x$ (when x is nonzero)

If the polar coordinates of a point are given, then the Cartesian coordinates are uniquely determined by equations 1 and 2. However, if the Cartesian coordinates of a point are given, the above equations do *not* determine unique polar coordinates. The origin must have $r = 0$, but it can have any angle as Θ. For any point other than the origin, there are two possible values of r, namely, $r = \pm(x^2 + y^2)^{1/2}$, and there are infinitely many choices for Θ.

Let us adopt conventions so that each point in the plane will have *unique* polar coordinates. There are several ways to do so.

First, we stipulate that the pole has Θ coordinate 0, i.e., it is represented by the ordered pair (0,0).

For the rest of the plane, we can adopt either of two approaches…

Approach #1:

Let (x,y) be any point other than the origin. Then it lies on a unique line L passing through the origin. Let Θ be the angle between L and the positive x axis, where $\Theta \in (-\pi/2, \pi/2]$. Specifically, if $x = 0$, then $\Theta = \pi/2$, and if x is nonzero, then $\Theta = \arctan(y/x)$. Let $r = (x^2 + y^2)^{1/2}$ if $x > 0$, or if $x = 0$ and $y > 0$. Let $r = -(x^2 + y^2)^{1/2}$ if $x < 0$, or if $x = 0$ and $y < 0$.

Under this approach, points in Quadrants I and IV and on the positive axes have positive r, while points in Quadrants II and III and on the negative axes have negative r. The value of r can be any real number, but the value of Θ always belongs to the interval $(-\pi/2, \pi/2]$. Note that $r = 0$ at one and only one point, the pole, and we have chosen to associate this point with a unique angle, $\Theta = 0$.

The entire plane is represented by the set of ordered pairs of real numbers $\{(r,\Theta) \mid r \in (-\infty,0) \cup (0,\infty), \Theta \in (-\pi/2, \pi/2]\} \cup \{(0,0)\}$. We have a one-to-one correspondence between points in the plane and the ordered pairs in this set.

Approach #2:

Let (x,y) be any point other than the origin.

If $x = 0$, then let $\Theta = \pi/2$ if $y > 0$ and let $\Theta = -\pi/2$ if $y < 0$.

If $x > 0$, then let $\Theta = \arctan(y/x)$.

If $x < 0$, then let $\Theta = \arctan(y/x) + \pi$.

Let $r = (x^2 + y^2)^{1/2}$.

Under this approach, all points other than the origin have positive r. The value of r can be any nonnegative real number, and the value of Θ always belongs to the interval $[-\pi/2, 3\pi/2)$.

The entire plane is represented by the set of ordered pairs of real numbers $\{(r,\Theta) \mid r \in (0,\infty), \Theta \in [-\pi/2, 3\pi/2)\} \cup \{(0,0)\}$. We have a one-to-one correspondence between points in the plane and the ordered pairs in this set.