15.8, Triple Integrals in Spherical Coordinates

Example:

Let E be the solid in the first octant (i.e., where $x \geq 0,\ \ y \geq 0,$ and $z \geq 0$) bounded by two spheres centered at the origin, the inner sphere with radius 1 and the outer sphere of radius 2. Find $\iiint_E (x^2 + y^2 + z^2)^{-3/2} dV.$

Note: If the integrand represents a density function, then this triple integral gives us the mass of the solid.

This problem is more easily solved if we use spherical coordinates. The integrand becomes $(\rho^2)^{-3/2}$, which simplifies to ρ^{-3}. dV becomes $\rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta$. The boundaries of integration are as follows: θ varies from 0 to $\frac{\pi}{2}$. φ varies from 0 to $\frac{\pi}{2}$. ρ varies from 1 to 2. Thus, we get:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^{-3}\ \rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^{-1} \sin\varphi\ d\rho\ d\varphi\ d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi \int_1^2 \rho^{-1} d\rho\ d\varphi\ d\theta$$

$$\int_1^2 \rho^{-1} d\rho = \ln|\rho| = \ln 2 - \ln 1 = \ln 2 - 0 = \ln 2.$$

Now we have $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi\ \ln 2\ d\varphi\ d\theta = \ln 2 \int_0^{\pi/2}\int_0^{\pi/2} \sin\varphi\ d\varphi\ d\theta$

$$\int_0^{\pi/2} \sin\varphi\ d\varphi = [-\cos\varphi]_0^{\pi/2} = [\cos\varphi]_{\pi/2}^0 = \cos 0 - \cos(\pi/2) = 1 - 0 = 1.$$

Now we have $\displaystyle\ln 2 \int_0^{\pi/2} d\theta = \ln 2[\theta]_0^{\pi/2} = \frac{\pi}{2}\ln 2$ or $\frac{\pi \ln 2}{2}$.

Digression: Before we consider our next example, let's digress for a moment. The equation $z^2 = x^2 + y^2$ is a circular cone whose top half lies above the x,y plane and whose bottom half lies below the x,y plane, along with the origin, $(0,0)$, which of course lies *in* the x,y plane. The equation of the top half is $z = \sqrt{x^2 + y^2}$, and the equation of the bottom half is $z = -\sqrt{x^2 + y^2}$. In cylindrical coordinates, the equation of the top half is $z = r$, and the equation of the bottom half is $z = -r$.

We may refer to the top half of the cone as an "upward-opening cone," and to the bottom half of the cone as a "downward-opening cone." (However, it would be more precise to say "upward-opening *half*-cone" and "downward-opening *half*-cone.") Thus, $z = r$ is an upward-opening cone and $z = -r$ is a downward-opening cone.

In spherical coordinates, the equation of the top half is $\rho \cos\varphi = \rho \sin\varphi$, or $\sin\varphi = \cos\varphi$, or $\tan\varphi = 1$, or $\varphi = \frac{\pi}{4}$. The equation of the bottom half is $\rho \cos\varphi = -\rho \sin\varphi$, or $-\sin\varphi = \cos\varphi$, or $\tan\varphi = -1$, or $\varphi = \frac{3\pi}{4}$.

Let us generalize: In spherical coordinates, the equation $\varphi = k$ is an upward-opening cone when $k \in (0, \frac{\pi}{2})$, and is a downward-opening cone when $k \in (\frac{\pi}{2}, \pi)$.

Example:

Let E be the region between the sphere $\rho = 1$ (which is the upper boundary surface) and the upward-opening cone $\varphi = \frac{\pi}{3}$ (which is the lower boundary surface). Find $\iiint_E (x^2 + y^2)dV$, using spherical coordinates.

We may write the integrand as r^2, but r is not a spherical coordinate, so we further rewrite it as $\rho^2 \sin^2\varphi$. We rewrite dV as $\rho^2 \sin\varphi \; d\rho \; d\varphi \; d\theta$. The boundaries of integration are as follows: θ varies from 0 to 2π. φ varies from 0 to $\frac{\pi}{3}$. ρ varies from 0 to 1. Thus, we get:

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin^2\varphi \; \rho^2 \sin\varphi \; d\rho \; d\varphi \; d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3\varphi \; d\rho \; d\varphi \; d\theta = \int_0^{2\pi} \int_0^{\pi/3} \sin^3\varphi \int_0^1 \rho^4 \; d\rho \; d\varphi \; d\theta$$

$$\int_0^1 \rho^4 \; d\rho = \tfrac{1}{5}[\rho^5]_0^1 = \tfrac{1}{5}$$

Now we have $\displaystyle\int_0^{2\pi} \int_0^{\pi/3} \tfrac{1}{5} \sin^3\varphi \; d\varphi \; d\theta = \tfrac{1}{5} \int_0^{2\pi} \int_0^{\pi/3} \sin^3\varphi \; d\varphi \; d\theta.$

$$\int_0^{\pi/3} \sin^3\varphi \; d\varphi = \int_0^{\pi/3} \sin^2\varphi \; \sin\varphi \; d\varphi = \int_0^{\pi/3} (1 - \cos^2\varphi) \; \sin\varphi \; d\varphi =$$

$$-1 \int_1^{1/2} (1 - u^2)du = \int_1^{1/2} (u^2 - 1)du = \left[\tfrac{1}{3}u^3 - u\right]_1^{1/2} = (\tfrac{1}{24} - \tfrac{1}{2}) - (\tfrac{1}{3} - 1) = \tfrac{5}{24}$$

Now we have $\displaystyle\tfrac{1}{5} \int_0^{2\pi} \tfrac{5}{24} \; d\theta = \tfrac{1}{24} \int_0^{2\pi} d\theta = \tfrac{1}{24}[\theta]_0^{2\pi} = \tfrac{1}{24}(2\pi) = \tfrac{\pi}{12}.$