Review for Final – Solutions

1. a) $\int x^3 \ln x\,dx$ $\quad\quad \ln x \quad x^3dx$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \frac{1}{x}dx \quad \frac{x^4}{4}$

$= \frac{x^4}{4}\ln x - \int \frac{x^3}{4}dx$

$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

b) $\int_0^1 \tan^{-1}x\,dx$ $\quad\quad \tan^{-1}x \quad 1\,dx$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \frac{1}{x^2+1}dx \quad x \quad\quad x\tan^{-1}x\Big|_0^1 = \frac{\pi}{4}$

$= \frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1}dx$ $\quad\quad u = x^2+1 \quad x=0 \;\; u=1$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad du = 2x\,dx \quad x=1 \;\; u=2$

$= \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{du}{u}$

$= \frac{\pi}{4} - \frac{1}{2}(\ln u)\Big|_1^2 = \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0) = \frac{\pi}{4} - \frac{1}{2}\ln 2$

$= \frac{\pi}{4} - \ln\sqrt{2}$

c) $\int x^2e^x dx$ $\quad\quad x^2 \quad e^x dx$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 2x\,dx \quad e^x$

$x^2e^x - \int 2xe^x dx$ $\quad\quad 2x \quad e^x dx$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 2dx \quad e^x$

$= x^2e^x - \left[2xe^x - \int 2e^x dx\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

2 a) $\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x (\tan x \sec x \, dx)$

$$= \int (\sec^2 x - 1)(\sec^2 x)(\tan x \sec x) dx$$

$$= \int (\sec^4 x - \sec^2 x)(\tan x \sec x) dx$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

b) $\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x (\sin x \, dx)$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \cos^2 x \sin x \, dx$$

$$= -\int (\cos^2 x - 2\cos^4 x + \cos^6 x)(-\sin x \, dx)$$

$$= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$$

c) $\int_0^\pi \sin^2 t \, dt = \int_0^\pi \left(\frac{1}{2} - \frac{\cos 2t}{2}\right) dt$

$$= \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^\pi$$

$$= \frac{\pi}{2}$$

3 a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

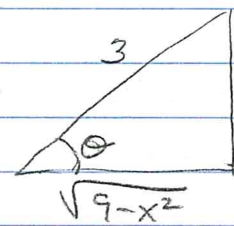
$9 - x^2 = 9 - 9\sin^2\theta$

$= 9(1-\sin^2\theta$

$= 9\cos^2\theta$

$\displaystyle\int \frac{3\cos\theta}{9\sin^2\theta}(3\cos\theta)\,d\theta$

$\displaystyle = \int \cot^2\theta\, d\theta$



$\displaystyle = \int (\csc^2\theta - 1)\,d\theta$

$= -\cot\theta - \theta + C$

$\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\displaystyle = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

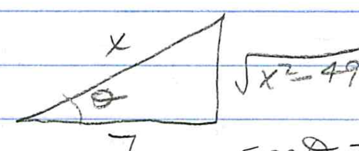
b) $\displaystyle\int \frac{1}{\sqrt{x^2-49}}$

$x = 7\sec\theta$

$49\sec^2\theta - 49 = 49\tan^2\theta$

$dx = 7\tan\theta\sec\theta\, d\theta$

$\displaystyle = \int \frac{7\tan\theta\sec\theta}{7\tan\theta}\,d\theta$



$\displaystyle = \int \sec\theta\, d\theta$

$\sec\theta = \frac{x}{7}$

$= \ln|\sec\theta + \tan\theta| + C$

$\tan\theta = \frac{\sqrt{x^2-49}}{7}$

$\displaystyle = \ln\left|\frac{x}{7} + \frac{\sqrt{x^2-49}}{7}\right| + C$

3 c) $\displaystyle\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

$x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = \frac{3\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$(4x^2+9)^{3/2} = \left(4\left(\frac{9}{4}\tan^2\right)+9\right)^{3/2}$

$= 27\sec^3\theta$

$$\int_0^{\frac{\pi}{3}} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\left(\frac{3}{2}\sec^2\theta\right)d\theta$$

$$= \int_0^{\pi/3} \frac{3}{16}\,\frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$\frac{\sin^3\theta}{\cos^{\cancel{3}}\theta} \cdot \frac{\cancel{\cos\theta}}{1}$

$$= \int_0^{\pi/3} \frac{3}{16}\sin^3\theta\cos^{-2}\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} (\sin\theta)\sin^2\theta\cos^{-2}\theta\,d\theta$$

$$= -\frac{3}{16}\int_0^{\pi/3} (-\sin\theta)(1-\cos^2\theta)\cos^{-2}\theta\,d\theta$$

$$= -\frac{3}{16}\int_0^{\pi/3} (-\sin\theta)(\cos^{-2}\theta - 1)\,d\theta$$

$$= -\frac{3}{16}\Big[-\cos^{-1}\theta - \cos\theta\Big]_0^{\pi/3}$$

$$= -\frac{3}{16}\left[-2 - \frac{1}{2} - (-1-1)\right]$$

$$= -\frac{3}{16}\left(-\frac{1}{2}\right) = \frac{3}{32}$$

4 a) $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

$$\frac{A}{x}+\frac{Bx+C}{x^2+4}=\frac{2x^2-x+4}{x(x^2+4)}$$

$$Ax^2+4A+Bx^2+Cx=2x^2-x+4$$

$$4A=4 \Rightarrow A=1$$

$$Ax^2+Bx^2=2x^2 \Rightarrow B=1$$

$$C=-1$$

$$\int \frac{1}{x}+\frac{x-1}{x^2+4}\,dx$$

$$=\int\left(\frac{1}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\right)dx$$

$$=\ln|x|+\frac{1}{2}\ln(x^2+4)-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

b) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

$$x^2+2x-1=A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

$$x^2+2x-1=2Ax^2+3Ax-2A+Bx^2+2Bx+2Cx^2-Cx$$

$-2A=-1 \Rightarrow A=\frac{1}{2}$ (constants)

$2A+B+2C=1$ (x^2 terms)

$3A+2B-C=2$ (x terms)

$1+B+2C=1$ (put in $A=\frac{1}{2}$)

$$B+2C=0$$

$$\frac{3}{2}+2B-C=2$$

$$2B-C=\frac{1}{2}$$

$\times 2$

$$4B-2C=1$$

$$B+2C=0$$

$$5B=1 \Rightarrow B=\frac{1}{5}$$

$$\frac{1}{5}+2C=0 \Rightarrow C=-\frac{1}{10}$$

$$\int\left(\frac{1}{2x}+\frac{1}{5(2x-1)}-\frac{1}{10(x+2)}\right)dx$$

$$\ln|2x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+C$$

5 a. $\displaystyle\int_{-\ln 2}^{\ln 2} \sqrt{1+(f'(x))^2}\, dx$

$f(x) = \frac{1}{2}(e^x + e^{-x})$

$f'(x) = \frac{1}{2}(e^x - e^{-x})$

$(f'(x))^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

$1+(f'(x))^2 = 1 + \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}$

$= \frac{1}{4}(e^{2x} + 2 + e^{-2x})$

$= \frac{1}{4}(e^x + e^{-x})^2$

$= \frac{1}{2}\displaystyle\int_{-\ln 2}^{\ln 2} (e^x + e^{-x})\, dx$

$= \frac{1}{2}(e^x - e^{-x})\Big|_{-\ln 2}^{\ln 2}$

$= \frac{1}{2}\left(2 - \frac{1}{2} - \left(\frac{1}{2} - 2\right)\right)$

$= \frac{3}{2}$

b) $y = \frac{1}{3}x^{3/2}$ $\quad 0 \leq x \leq 60$

$y' = \frac{1}{2}x^{\frac{1}{2}}$

$(y')^2 = \frac{1}{4}x$

$\sqrt{1+\frac{1}{4}x} = \frac{1}{2}\sqrt{4+x}$

$\frac{1}{2}\displaystyle\int_0^{60} (4+x)^{\frac{1}{2}}\, dx$

$= \frac{1}{2}\left[\frac{2(4+x)^{3/2}}{3}\right]_0^{60}$

$= \frac{1}{3}\left[8^3 - 8\right]$

$= \frac{504}{3}$

$= 168$

6 $25 = \frac{1}{4}k \Rightarrow k = 100$

a) $\int_0^2 100x\,dx = 50x^2\Big|_0^2 = 200\text{ J}$

b) $\int_{1.5}^{2.5} 100x\,dx = 50x^2\Big|_{1.5}^{2.5} = 312.5 - 112.5 = 200\text{J}$

7 a) $\lim_{k\to\infty} \frac{k^2-1}{k^2+1} = \lim_{k\to\infty} \frac{1-\frac{1}{k^2}}{1+\frac{1}{k^2}} = 1 \neq 0$ so diverges (divergent test)

b) $\lim_{n\to\infty} \frac{(n+1)^{100}100^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{100}100^n}$

$= \lim_{n\to\infty} \frac{(n+1)^{100}\cancel{100^n}\cdot 100}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{n^{100}\cancel{100^n}}$

$= \lim_{n\to\infty} \frac{\cancel{(n+1)^{100}}^{99}\,100}{\cancel{(n+1)}n^{100}}$

$= 100 \lim_{n\to\infty} \frac{(n+1)^{99}}{n(n)^{99}}$

$= 100 \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{99}\frac{1}{n}$

$= 100 \lim_{n\to\infty} \underset{\uparrow\,0}{\frac{1}{n}}\left(1+\frac{1}{n}\right)^{99}$ $\quad\left(1+\frac{1}{n}\right)^{99} < e$ when $n > 99$

$= 0 < 1$ so converges

c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} = \sum_{n=1}^{\infty}\left(\frac{\sqrt{n}}{n^2} + \frac{4}{n^2}\right)$

$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + 4\sum_{n=1}^{\infty}\frac{1}{n^4}$ both p-series with $p > 1$ so converges

8 a) $\lim_{k\to\infty} \frac{1}{\ln k} = 0$ so alt. series converges

for $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ $\quad k < e^k \Rightarrow \ln k < k \Rightarrow \frac{1}{\ln k} > \frac{1}{k}$

$\sum \frac{1}{k}$ diverges (harmonic)

so $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ also diverges

so converges conditionally

b) $\sum_{k=1}^{\infty} \frac{(-9)^k}{k(10)^{k+1}}$ $\quad \lim_{k\to\infty} \frac{9^k}{k(10)^{k+1}} = \lim_{k\to\infty} \frac{1}{10k}\left(\frac{9}{10}\right)^k = 0$

so alt. series converges

$\lim_{k\to\infty} \frac{9^{k+1}}{(k+1)10^{k+2}} \cdot \frac{k10^{k+1}}{9^k}$

$= \lim_{k\to\infty} \left(\frac{9}{10}\right)\left(\frac{k}{k+1}\right) = \frac{9}{10} \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}}\right) = \frac{9}{10} < 1$

so absolutely convergent

c) $\sum_{k=1}^{\infty} (-1)^{k-1} k^{-\frac{1}{3}}$ $\quad \lim_{k\to\infty} \frac{1}{k^{\frac{1}{3}}} = 0$ so alt. series converges.

$\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{3}}}$ p-series with $k < 1$ so diverges

conditionally convergent

9. $x = r(\theta - \sin\theta) \qquad y = r(1-\cos\theta)$

a) $\frac{dy}{d\theta} = r\sin\theta \qquad \frac{dx}{d\theta} = r - r\cos\theta = r(1-\cos\theta)$

$\frac{dy}{dx} = \frac{r\sin\theta}{r(1-\cos\theta)} \qquad \text{at } \theta = \frac{\pi}{3} = \frac{r\left(\frac{\sqrt{3}}{2}\right)}{r\left(\frac{1}{2}\right)} = \sqrt{3}$

$x = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(\frac{1}{2}\right)$

$y - \frac{r}{2} = \sqrt{3}\left(x - r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$

b) $r(1-\cos\theta) = 0 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

c) $r\sin\theta = 0 \Rightarrow \sin\theta = 0 \qquad \theta = \cancel{0}, \pi$

10 a) $(3\sqrt{3}, 3) \qquad r = \sqrt{(3\sqrt{3})^2 + 3^2} = 6$

$\tan\theta = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$\left(6, \frac{\pi}{6}\right)$

b) $(-1, \sqrt{3}) \qquad r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \qquad \theta$ in Q II so $\theta = \frac{2\pi}{3}$

$\left(2, \frac{2\pi}{3}\right)$

11 a) $\left(2, -\frac{2\pi}{3}\right) \qquad x = 2\cos\left(\frac{-2\pi}{3}\right) = 2\left(\frac{-1}{2}\right) = -1$

$y = 2\sin\left(\frac{-2\pi}{3}\right) = 2\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{3}$

$(-1, -\sqrt{3})$

b) $\left(-2, \frac{3\pi}{4}\right) \qquad x = -2\cos\left(\frac{3\pi}{4}\right) = -2\left(\frac{-\sqrt{2}}{2}\right) = \sqrt{2}$

$y = -2\sin\left(\frac{3\pi}{4}\right) = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

$(\sqrt{2}, -\sqrt{2})$

12. a) $\int_0^{2\pi} \frac{1}{2}(2+2\cos\theta)^2 \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (4+8\cos\theta+4\cos^2\theta)\, d\theta$

$= 2\int_0^{2\pi} \left(1+2\cos\theta + \frac{\cos 2\theta + 1}{2}\right) d\theta$

$= 2\int_0^{2\pi} \frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}) \, d\theta$

$= 2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= 2\left[\frac{3}{2}(2\pi)\right]$

$= 6\pi$

b) $4x = x\sqrt{25-x^2}$

$16x^2 = x^2(25-x^2)$

$0 = 9x^2 - x^4$

$0 = x^2(9-x^2)$

$x=0 \quad x=\pm 3$ 1^{st} quadrant so 3

$\int_0^3 (x\sqrt{25-x^2} - 4x)\, dx$

$= \int_0^3 x\sqrt{25-x^2}\, dx - 4\int_0^3 x \, dx$ $u = 25-x^2$

$du = -2x\,dx$

$= -\frac{1}{2}\int_0^3 (-2x)(25-x^2)^{\frac{1}{2}}\, dx - \frac{4x^2}{2}\Big|_0^3$

$= -\frac{1}{2}(25-x^2)^{3/2}\left(\frac{2}{3}\right)\Big|_0^3 - 18$

$= -\frac{1}{3}(64-125) - 18$

$= \frac{7}{3}$

12 c) $f(x)=x$ and $g(x)=x^2$ intersect at $x=1$ so the graphs cross at that point

$$\int_0^1 (x-x^2)\,dx + \int_1^2 (x^2-x)\,dx$$

$$= \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2}-\frac{1}{3}+\frac{8}{3}-2-\frac{1}{3}+\frac{1}{2}$$

$$= 1$$

13 a) $y=\sec x \quad y=2 \quad 0\le x\le \frac{\pi}{3}$

$$\int_0^{\pi/3} \pi\left(2^2-\sec^2 x\right)dx$$

$$= \pi\left[4x-\tan x\right]_0^{\pi/3}$$

$$= \pi\left[\frac{4\pi}{3}-\sqrt{3}\right] = \frac{4\pi^2}{3}-\sqrt{3}\,\pi$$

b) $y=(1-x^2)^{-1/2} \quad \left[0,\frac{\sqrt{3}}{2}\right]$ shell

$$\int_0^{\sqrt{3}/2} 2\pi x\left(1-x^2\right)^{-\frac{1}{2}}dx \qquad u=1-x^2 \quad du=-2x\,dx$$

$$= -\pi\int \underbrace{(-2x)}_{du}\underbrace{\left(1-x^2\right)^{-1/2}}_{u^{-\frac{1}{2}}}dx$$

$$= -\pi\left[\left(1-x^2\right)^{\frac{1}{2}}(2)\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= -\pi\left[\frac{1}{2}(2)-1(2)\right]$$

$$= \pi$$

14 a) $\int_0^2 2\pi\left(\frac{1}{3}x^3\right)\sqrt{1+x^4}\,dx$ $\qquad f(x)=\frac{1}{3}x^3$

$f'(x)=x^2$

$= \frac{2\pi}{3}\int_0^2 x^3\sqrt{1+x^4}\,dx$ $\qquad u=1+x^4$

$du = 4x^3dx$

$x=0 \Rightarrow u=1$

$x=2 \Rightarrow u=17$

$\frac{2\pi}{3}\left(\frac{1}{4}\right)\int_1^{17} u^{\frac{1}{2}}\,du$

$= \frac{\pi}{6}\left[\frac{2u^{3/2}}{3}\right]_1^{17}$

$= \frac{\pi}{9}\left[17^{2/3}-1\right]$

b) $\pi\int_0^2 \frac{1}{9}x^6\,dx = \pi\left[\frac{x^7}{63}\right]_0^2 = \frac{128\pi}{63}$

15 $f(x)=\sqrt{x}$ $\quad a=4$

$f(4)=2$

$f'(x)=\frac{1}{2\sqrt{x}}$ $\qquad f'(4)=\frac{1}{4}$

$f''(x)=\frac{-1}{4x^{3/2}}$ $\qquad f''(4)=\frac{-1}{32}$

$p_2(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$

16. $f(x)=\frac{2}{(1-2x)^2}$ $\qquad \frac{1}{1-x}=\sum_{k=0}^{\infty} x^k$ (geometric) and converges for $|x|<1$

$\frac{1}{1-2x}=\sum_{k=0}^{\infty}(2x)^k$ converge: $|x|<\frac{1}{2}$

$\frac{d}{dx}\left(\frac{1}{1-2x}\right)=\frac{2}{(1-2x)^2}=\sum_{k=1}^{\infty}k(2x)^{k-1}(2)$

$=\sum_{k=1}^{\infty}2k(2x)^{k-1}$