3.10 Linear Approximations and Differentials

So far we have seen very difficult functions that have given us problems when trying to find specific y values for different parts of their domains. In the past we have seen that if we zoom in "enough" to a specific point on the function, the graph of the function looks like a tangent line. For that reason we could say that we can find the approximate y-value of a function that is **close** to $x = a$ by using the tangent line equation for point $(a, f(a))$. The equation for this tangent line is:

$\boldsymbol{y - f(a) = f'(a)(x-a)}$ or $\boldsymbol{y = f(a) + f'(a)(x-a)}$ and the approximation $\boldsymbol{y \approx f(a) + f'(a)(x-a)}$ is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is the tangent line, that is $\boldsymbol{L(x) = f(a) + f'(a)(x-a)}$ is called the ***linearization of f at a***.

Definition: Linear Approximation to f at a.

Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function $\boldsymbol{L(x) = f(a) + f'(a)(x-a)}$, for x in L.

Example: Find the linear approximation to $f(x) = \sin(x)$ and $x = 0$ and use it to approximate $\sin(2.5°)$.

We need to use $L(x) = f(a) + f'(a)(x - a)$.

$f(x) = \sin x$ and $x = 0 \Rightarrow f(0) = \sin 0 = 0 \quad f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$
L(x) = 0 + 1(x – 0)
L(x) = x (this is the linear approximation)

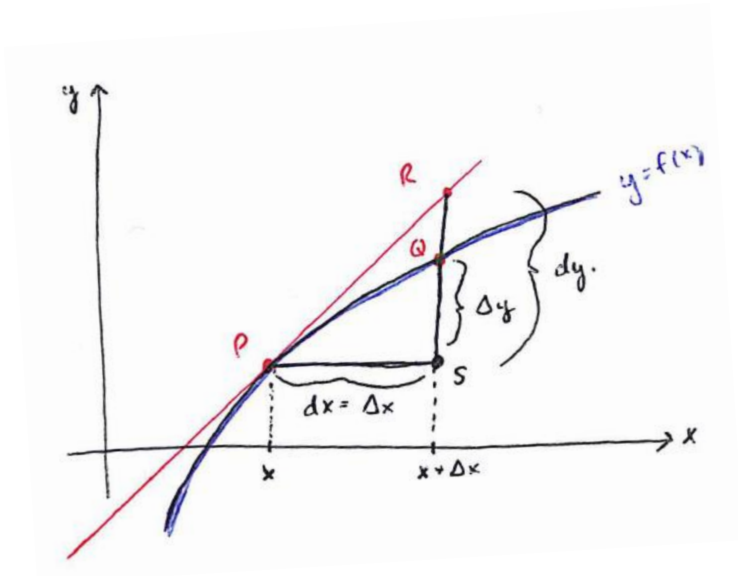
Now convert 2.5° to radians. $\Rightarrow 2.5° \cdot \frac{\pi}{180} \approx 0.04363$ radians
Therefore, $\sin(2.5°) \approx L(0.04363 \approx \mathbf{0.04363}$. Now use your calculator and approximate sin(2.5°). How close is it to the linear approximation?

Example: Find the linear approximation function L to the function at point a given $f(x) = e^{-x}$, and $a = \ln(2)$.

Use $\boldsymbol{L(x) = f(a) + f'(a)(x-a)}$ $\qquad f(a) = e^{-\ln 2} = \frac{1}{e^{\ln 2}} = \frac{1}{2} \qquad f'(a) = -e^{\ln 2} = -\frac{1}{e^{\ln 2}} = -\frac{1}{2}$
$L(x) = f(a) + f'(a)(x-a) \Rightarrow \boldsymbol{L(x) = \frac{1}{2} - \frac{1}{2}(x - \ln 2)}$

DIFFERENTIALS

We can also express a linear approximation in the terminology and notation of differentials. If $y = f(x)$ is a differential, then we could say:
$\frac{dy}{dx} = f'(x)$ assuming $dx \neq 0$
$dy = f'(x) \cdot dx$ (This is called the differential) Notice we can determine the value of dy if we are given dx and x, where x is a value in the domain of f. Graphically, this is what we have:



Notice the corresponding change in **y** is
$\Delta y = f(x + \Delta x) - f(x)$
dy represents the amount that the tangent line rises or falls, whereas Δy represents the amount that the **curve**, ***y = f(x)*** rises or falls when $\boldsymbol{x}$ changes by an amount ***dx***.
$\boldsymbol{dx}$ always = $\Delta \boldsymbol{x}$ **but** $\boldsymbol{dy}$ is just an approximation of Δy.

Example: Find the differential ***dy*** and evaluate ***dy*** for the given values of $\boldsymbol{x}$ and ***dx.***

$y = e^{\frac{x}{10}},\ x = 0,\ \ dx = 0.1$

We have that the differential is ***dy = f '(x) · dx***

$f'(x) = e^{\frac{x}{10}} \cdot \frac{1}{10} = \frac{e^{\frac{x}{10}}}{10}$

Thus $\boldsymbol{dy} = \frac{\boldsymbol{e^{\frac{x}{10}}}}{\boldsymbol{10}} \cdot \boldsymbol{dx}$ This is the differential, now evaluate ***dy***.

$dy = \frac{e^{\frac{0}{10}}}{10}(0.1) = \frac{\mathbf{1}}{\mathbf{100}}$

Example: Use the notation of differentials to write the approximate change in ***f(x) = 3cos²(x)*** given a small change ***dx***.

Find the differential ***dy***, ***dy = f '(x)dx***
$f'(x) = 3 \cdot 2\cos x \cdot -\sin x$
$\quad = -3 \cdot 2\sin x \cos x$ (Use the double angle identity)
$\quad = -3\sin 2x$
Thus $\boldsymbol{dy} = \mathbf{-3\,sin}(\boldsymbol{2x})\boldsymbol{dx}$

The interpretation is that a small change ***dx*** in the independent variable $\boldsymbol{x}$ produces an approximate change in the dependent variable of $\boldsymbol{dy} = \mathbf{-3\,sin}(\boldsymbol{2x})\boldsymbol{dx}$ in ***y***.